Invariant totally geodesic unit vector fields on three-dimensional Lie groups.

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Abstract

We give a complete list of those left invariant unit vector fields on three-dimensional Lie groups with the left-invariant metric that generate a totally geodesic submanifold in the unit tangent bundle of a group with the Sasaki metric. As a result, each class of three-dimensional Lie groups admits the totally geodesic unit vector field. From geometrical viewpoint, the field is either parallel or characteristic vector field of a natural almost contact structure on the group.

Key words: *Sasaki metric, totally geodesic unit vector field, almost contact structure, Sasakian structure.*

AMS subject class: Primary 53B20, 53B25; Secondary 53C25.

Introduction

Let (M^n, g) be Riemannian manifold and (T_1M^n, g_s) its unit tangent bundle with Sasaki metric. Consider a unit vector field ξ as a (local) mapping

$$\xi : M^n \to T_1M^n.$$

Definition 1 *A unit vector field ξ on Riemannian manifold M^n is called totally geodesic if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is totally geodesic submanifold in the unit tangent bundle T_1M^n with Sasaki metric.*

In a similar way one can define a *locally minimal* unit vector field as the field of *zero mean curvature.* A number of examples of locally minimal unit vector fields was found recently by L. Vanhecke, E. Boeckx, K. Tsukada, J.C. González -Dávila, O. Gil-Medrano and others [3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16]. Particulary, K Tsukada and L. Vanhecke [15] described all minimal left-invariant unit vector fields on three-dimensional Lie groups with the left-invariant metric.

The key step to the totally geodesic unit vector fields was made in [18], where the author have found the second fundamental form of $\xi(M^n)$ explicitly, using a special normal frame. This expression allowed, also, to find an

examples of unit vector fields of *constant mean curvature*. Using this expression, the author described all the 2-manifolds that admit a totally geodesic unit vector field and the field itself [22]. In the case of higher dimensions only partial results are known. The most general states that if M^{2m+1} is a Sasakian manifold and ξ is a characteristic vector field of the Sasakian structure, then $\xi(M^{2m+1})$ is totally geodesic in T_1M^{2m+1} [20].

Particularly, the Hopf unit vector field on a unit S^{2m+1} is totally geodesic. More specifically, the Hopf vector field belongs to the class of left invariant unit vector fields on S^3 as a Lie group with the left-invariant Riemannian metric. In this paper, we give full description of 3-dimensional Lie groups with left-invariant metric which admit a totally geodesic left-invariant unit vector fields and the fields themselves. As a consequence we have found that, in non-trivial case, for each totally geodesic left invariant unit vector field ξ the structure $\big(\phi = -\nabla\xi,\ \xi,\ \eta = \langle\xi,\cdot\rangle\big)$ is an almost contact one on the corresponding Lie group and ξ is a characteristic vector field of this structure. If ξ is a Killing unit vector field, then the structure is Sasakian.

The paper is organized as follows. In Section 1 we give some preliminaries. In Section 2 we consider the unimodular Lie groups. We prove that if the totally geodesic unit vector field exists on a given group, then it is an eigenvector of the Ricci tensor which corresponds to the Ricci principal curvature $\rho = 2$ (Theorem 2.1). The Theorem 2.2 provides the complete list of totally geodesic unit vector fields on a corresponding Lie group as well as the conditions on structure constants of the group. In a series of Propositions 2.2 – 2.6, we give a description of totally geodesic unit vector field in unimodular case from the contact geometry viewpoint.

In Section 3 we consider the non-unimodular case. The Theorem 3.2 provides an explicit expression for the totally geodesic unit vector field as well as the conditions on structure constants of the corresponding group. Finally, the Proposition 3.1 gives the geometrical characterization of the totally geodesic unit vector field and clarifies the structure of the corresponding non-unimodular Lie group.

1 Some preliminaries

Let (M^n, g) be Riemannian manifold. Denote by ∇ the Levi-Civita connection on M^n. Introduce a pointwise linear operator $A_\xi : T_qM^n \to \xi_q^\perp$, acting as

$$A_\xi X = -\nabla_X \xi.$$

In case of integrable distribution $\xi^\perp$, the unit vector field ξ is called *holonomic*. In this case the operator A_ξ is symmetric and is known as Weingarten or a *shape operator* for each hypersurface of the foliation. In general, A_ξ is not symmetric but formally preserves the Codazzi equation. Namely, a

covariant derivative of A_ξ is defined by

$$(\nabla_X A_\xi)Y = -\nabla_X\nabla_Y\xi + \nabla_{\nabla_X Y}\xi. \tag{1}$$

Then for the curvature operator of M^n we can write down the Codazzi-type equation

$$R(X,Y)\xi = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y.$$

From this viewpoint, it is natural to call the operator A_ξ by *non-holonomic shape* operator.

Introduce a symmetric tensor field

$$Hess_\xi(X,Y) = \frac{1}{2}\big[(\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y\big], \tag{2}$$

which is a *symmetric part* of covariant derivative of A_ξ. The trace

$$-\sum_{i=1}^{n} Hess_\xi(e_i,e_i) := \Delta\xi,$$

where $e_1,\dots e_n$ is an orthonormal frame, is known as *rough Laplacian* [1] of the field ξ. Therefore, one can treat the tensor field (2) as a *rough Hessian* of the field.

For the mapping $f:(M,g)\to(N,h)$ between Riemannian manifolds the *energy* of f is defined as

$$E(f) := \frac{1}{2}\int_M |d\,f|^2\,dVol_M,$$

where $|d\,f|$ is a norm of 1-form $d\,f$ in the cotangent bundle T^*M. The mapping f is called harmonic if it is a critical point of the functional $E(f)$. Supposing on T_1M the Sasaki metric, a unit vector field is called *harmonic*, if it is a critical point of energy functional of mapping $\xi: M^n\to T_1M^n$. This definition presumes the variation within the class of unit vector fields. From this viewpoint, the unit vector field is harmonic if and only if [17]

$$\Delta\xi = -|\nabla\xi|^2\xi.$$

There exist the unit vector fields that fail to be critical within a wider class of all mappings $f: M^n\to T_1M^n$ [7]. Introduce a tensor field

$$Hm_\xi(X,Y) = \frac{1}{2}\big[R(\xi,A_\xi X)Y + R(\xi,A_\xi Y)X\big]. \tag{3}$$

A *harmonic* unit vector field ξ defines a *harmonic mapping* $\xi: M^n\to T_1M^n$ if and only if [7]

$$\sum_{i=1}^{n} Hm_\xi(e_i,e_i) = 0.$$

From this viewpoint, it is natural to call the tensor field (3) by *harmonicity tensor* of the field ξ.

In terms of the tensors $Hess_\xi$ and Hm_ξ the conditions on ξ to be totally geodesic are as follows [23].

Theorem 1.1 *A unit vector field ξ on a given Riemannian manifold M^n is totally geodesic if and only if*

$$Hess_\xi(X,Y) + A_\xi Hm_\xi(X,Y) - \big\langle A_\xi X, A_\xi Y\big\rangle \xi = 0$$

for all vector fields X, Y on M^n.

It is natural to introduce a tensor field

$$TG_\xi(X,Y) = Hess_\xi(X,Y) + A_\xi Hm_\xi(X,Y) - \big\langle A_\xi X, A_\xi Y\big\rangle \xi \qquad (4)$$

as a *total geodesity* tensor field.

The treatment of 3-dimensional Lie groups is based on J. Milnor description of 3-dimensional Lie groups via the structure constants [13].

In the case of *unimodular* Lie group with the left-invariant metric, there is an orthonormal frame e_1, e_2, e_3 of its Lie algebra such that the bracket operations are defined by

$$[e_2,e_3] = \lambda_1 e_1, \quad [e_3,e_1] = \lambda_2 e_2, \quad [e_1,e_2] = \lambda_3 e_3. \qquad (5)$$

The constants $\lambda_1, \lambda_2, \lambda_3$ completely determine the topological structure of corresponding Lie group as in the following table:

signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie grous
$+,+,+$	$SU(2)$ or $SO(3)$
$+,+,-$	$SL(2,\mathbb{R})$ or $O(1,2)$
$+,+,0$	$E(2)$
$+,-,0$	$E(1,1)$
$+,0,0$	Nil^3 (Heisenberg group)
$0,0,0$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$

In the case of non-unimodular Lie group, let e_1 be a unit vector orthogonal to the unimodular kernel U and choose an orthonormal basis $\{e_2, e_3\}$ of U which diagonalizes the symmetric part of $ad_{e_1}\big|_U$. Then the bracket operation can be expressed as

$$[e_1,e_2] = \alpha\, e_2 + \beta\, e_3, \quad [e_1,e_3] = -\beta\, e_2 + \delta\, e_3, \quad [e_2,e_3] = 0. \qquad (6)$$

If necessary, changing e_1 to $-e_1$, we can assume $\alpha + \delta > 0$ and by possibly alternating e_2 and e_3, we may also suppose $\alpha \geq \delta$ [15].

2 The unimodular case

Choose the orthonormal frame as in (5). Define a connection numbers by

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i.$$

Then the Levi-Civita covariant derivatives can be expressed via the cross-products as follows

$$\nabla_{e_i} e_k = \mu_i \, e_i \times e_k. \tag{7}$$

For any left-invariant unit vector field $\xi = x_1 e_1 + x_2 e_2 + x_3 e_3$ we have

$$\nabla_{e_i} \xi = \mu_i \, e_i \times \xi. \tag{8}$$

Denote $N_i = e_i \times \xi$. Then

$$\nabla_{e_i} \xi = \mu_i \, e_i \times \xi = \mu_i \, N_i. \tag{9}$$

As a consequence, the matrix of the Weingarten operator takes the form

$$A_\xi = \begin{pmatrix} 0 & -\mu_2 x_3 & \mu_3 x_2 \\ \mu_1 x_3 & 0 & -\mu_3 x_1 \\ -\mu_1 x_2 & \mu_2 x_1 & 0 \end{pmatrix} \tag{10}$$

We will need the following technical Lemma.

Lemma 2.1 *Let G be a three-dimensional unimodular Lie group with the left-invariant metric and let $\{e_i,\ i = 1, 2, 3\}$ be an orthonormal basis for the Lie algebra satisfying (5). Then for any left-invariant unit vector field $\xi = x_1 e_1 + x_2 e_2 + x_3 e_3$ we have*

$$\begin{aligned} & A_\xi e_i = -\mu_i \, e_i \times \xi = -\mu_i \, N_i \\ & (\nabla_{e_i} A_\xi) e_i = \mu_i^2 (\xi - x_i e_i), \\ & (\nabla_{e_i} A_\xi) e_k = \varepsilon_{ikm} \mu_i \mu_m N_m - \mu_i \mu_k x_i e_k \quad (i \neq k), \\ & R(e_i, e_k)\xi = -\varepsilon_{ikm} \sigma_{ik} N_m, \end{aligned}$$

where $\sigma_{ik} = \sigma_{ki} = \mu_i \mu_m + \mu_k \mu_m - \mu_i \mu_k$ and $\varepsilon_{ikm} = \langle e_i \times e_k, e_m \rangle$.

Proof. The first equality comes from definitions. For the rest, we have

$$\begin{aligned} & \nabla_{e_i} e_k = \mu_i \, e_i \times e_k = \varepsilon_{ikm} \mu_i \, e_m, \quad \nabla_{\nabla_{e_i} e_k} \xi = \varepsilon_{ikm} \mu_i \mu_m N_m, \\ & \nabla_{e_i} \nabla_{e_k} \xi = \mu_i \mu_k \, e_i \times (e_k \times \xi) = \mu_i \mu_k (x_i e_k - \delta_{ik} \xi). \end{aligned}$$

Therefore,

$$(\nabla_{e_i} A_\xi) e_k = \nabla_{\nabla_{e_i} e_k} \xi - \nabla_{e_i} \nabla_{e_k} \xi = \varepsilon_{ikm} \mu_i \mu_m N_m + \mu_i \mu_k (\delta_{ik} \xi - x_i e_k)$$

Setting $i=k$ and then $i\neq k$, we get the second and the third equalities. From Codazzi equation

$$\begin{aligned} R(e_i,e_k)\xi = \;& (\nabla_{e_k}A_\xi)e_i-(\nabla_{e_i}A_\xi)e_k = \\ & \varepsilon_{kim}\mu_k\mu_m N_m+\mu_k\mu_i(\delta_{ki}\xi-x_ke_i)- \\ & \varepsilon_{ikm}\mu_i\mu_m N_m-\mu_i\mu_k(\delta_{ik}\xi-x_ie_k) = \\ & -\varepsilon_{ikm}(\mu_i\mu_m+\mu_k\mu_m)N_m+\mu_i\mu_k(x_ie_k-x_ke_i). \end{aligned}$$

Remark, that $N_m=\varepsilon_{ikm}(x_ie_k-x_ke_i)$ and hence

$$R(e_i,e_k)\xi=-\varepsilon_{ikm}(\mu_i\mu_m+\mu_k\mu_m-\mu_i\mu_k)N_m$$

■

Remark that chosen frame diagonalises the Ricci tensor [13]. Moreover,

$$2\mu_i\mu_k=\rho_m,$$

where ρ_m is the principal Ricci curvature and $i\neq k\neq m$. It also worthwhile to mention that

$$\sigma_{ik}=\frac{1}{2}(\rho_k+\rho_i-\rho_m)$$

is nothing else bur the sectional curvature of the left-invariant metric in a direction of $e_i\wedge e_k$.

Lemma 2.2 *Let G be a three-dimensional unimodular Lie group with the left-invariant metric and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5)*. Then a left-invariant unit vector field $\xi=x_1e_1+x_2e_2+x_3e_3$ is totally geodesic if and only if for any $i\neq k\neq m$*

$$\begin{aligned} TG(e_i,e_i) = \;& x_i\mu_i\Big\{x_m(\sigma_{ik}\mu_k-\mu_i)N_k-x_k(\sigma_{im}\mu_m-\mu_i)N_m\Big\}=0, \\ 2TG(e_i,e_k) = \;& \varepsilon_{ikm}\Big\{-x_ix_m\mu_i(\sigma_{ik}\mu_i-\mu_k)N_i+x_kx_m\mu_k(\sigma_{ik}\mu_k-\mu_i)N_k+ \\ & \Big(\mu_i\mu_m(1-\sigma_{km})-\mu_k\mu_m(1-\sigma_{im})+\mu_i(\sigma_{km}\mu_m-\mu_k)x_i^2- \\ & \mu_k(\sigma_{im}\mu_m-\mu_i)x_k^2\Big)N_m\Big\}=0, \end{aligned}$$

where $\sigma_{ik}=\sigma_{ki}=\mu_i\mu_m+\mu_k\mu_m-\mu_i\mu_k$ and $\varepsilon_{ikm}=\langle e_i\times e_k,e_m\rangle$.

Proof. Calculate $Hess_\xi(e_i,e_i)-|A_\xi e_i|^2\,\xi$. We have

$$\begin{aligned} (\nabla_{e_i}A_\xi)e_i-|A_\xi e_i|^2\xi = \;& \mu_i^2(\xi-x_ie_i)-\mu_i^2(1-x_i^2)\xi=-\mu_i^2x_i(e_i-x_i\xi)= \\ & -\mu_i^2x_i((1-x_i^2)e_i-x_ix_ke_k-x_ix_me_m)= \\ & -\mu_i^2x_i((x_k^2+x_m^2)e_i-x_ix_ke_k-x_ix_me_m)= \\ & -\mu_i^2x_i(x_k(x_ke_i-x_ie_k)+x_m(x_me_i-x_ie_m)= \\ & -\mu_i^2x_i(x_k\varepsilon_{kim}N_m+x_m\varepsilon_{mik}N_k)= \\ & \varepsilon_{ikm}\mu_i^2x_i(x_kN_m-x_mN_k). \end{aligned} \tag{11}$$

Find now $A_\xi Hm_\xi(e_i,e_i)$. Using Lemma 2.1, we have

$$\begin{aligned}
Hm_\xi(e_i,e_i) = R(\xi, A_\xi e_i)e_i = \\
&\langle R(\xi, A_\xi e_i)e_i, e_k\rangle e_k + \langle R(\xi, A_\xi e_i)e_i, e_m\rangle e_m = \\
&\langle R(e_i, e_k)\xi, A_\xi e_i\rangle e_k + \langle R(e_i, e_m)\xi, A_\xi e_i\rangle e_m = \\
\mu_i\Big(&\varepsilon_{ikm}\sigma_{ik}\langle e_m\times\xi, e_i\times\xi\rangle e_k + \varepsilon_{imk}\sigma_{im}\langle e_k\times\xi, e_i\times\xi,\rangle e_m\Big) = \\
&-\mu_i\varepsilon_{ikm}\Big(x_ix_m\sigma_{ik}e_k - x_ix_k\sigma_{im}e_m\Big)
\end{aligned}$$

Therefore,

$$A_\xi Hm_\xi(e_i,e_i) = \varepsilon_{ikm}\mu_ix_i\Big(x_m\sigma_{ik}\mu_kN_k - x_k\sigma_{im}\mu_mN_m\Big) \tag{12}$$

Adding (11) and (12), after evident simplifications we get $TG_\xi(e_i,e_i)$.

Applying Lemma 2.1 for $i\neq k$, we get

$$\begin{aligned}
2Hess_\xi(e_i,e_k) = \ & (\nabla_{e_i}A_\xi)e_k + (\nabla_{e_k}A_\xi)e_i = \\
& \varepsilon_{ikm}(\mu_i\mu_m - \mu_k\mu_m)N_m - \mu_i\mu_k(x_ie_k + x_ke_i).
\end{aligned}$$

Evidently,

$$\langle A_\xi e_i, A_\xi e_k\rangle\xi = \mu_i\mu_k\langle e_i\times\xi, e_k\times\xi\rangle\xi = -\mu_i\mu_kx_ix_k\xi$$

Subtracting, we get

$$\begin{aligned}
2Hess_\xi(e_i,e_k) - 2\langle A_\xi e_i, A_\xi e_k\rangle\xi = \varepsilon_{ikm}(\mu_i\mu_m - \mu_k\mu_m)N_m - \\
\mu_i\mu_k(x_ie_k + x_ke_i - 2x_ix_k\xi).
\end{aligned}$$

Observe that

$$\begin{aligned}
x_ie_k + x_ke_i - 2x_ix_k\xi = x_i(1-2x_k^2)e_k + x_k(1-2x_i^2)e_i - 2x_ix_kx_me_m = \\
x_k(x_k^2 - x_i^2 + x_m^2)e_i + x_i(-x_k^2 + x_i^2 + x_m^2)e_k - 2x_ix_kx_me_m = \\
x_kx_m(x_me_i - x_ie_m) + x_ix_m(x_me_k - x_ke_m) + (x_k^2 - x_i^2)(x_ke_i - x_ie_k) = \\
x_kx_m\varepsilon_{mik}e_k\times\xi + x_ix_m\varepsilon_{mki}e_i\times\xi + (x_k^2 - x_i^2)\varepsilon_{kim}e_m\times\xi = \\
\varepsilon_{ikm}\big(x_kx_mN_k - x_ix_mN_i - (x_k^2 - x_i^2)N_m\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
2Hess_\xi(e_i,e_k) - 2\langle A_\xi e_i, A_\xi e_k\rangle\xi = \\
\varepsilon_{ikm}\Big\{\mu_i\mu_kx_m(-x_kN_k + x_iN_i) + (\mu_i\mu_m - \mu_k\mu_m + (x_k^2 - x_i^2)\mu_i\mu_k)N_m\Big\}
\end{aligned}$$

To find $Hm_\xi(e_i,e_k)$, calculate $R(\xi, A_\xi e_i)e_k$. We have

$$\begin{aligned}
R(\xi, A_\xi e_i)e_k = \ & \langle R(\xi, A_\xi e_i)e_k, e_i\rangle e_i + \langle R(\xi, A_\xi e_i)e_k, e_m\rangle e_m = \\
& \langle R(e_k, e_i)\xi, A_\xi e_i\rangle e_i + \langle R(e_k, e_m)\xi, A_\xi e_i\rangle e_m = \\
& \mu_i\sigma_{ki}\varepsilon_{kim}\langle e_m\times\xi, e_i\times\xi\rangle e_i + \mu_i\sigma_{km}\varepsilon_{kmi}\langle e_i\times\xi, e_i\times\xi\rangle e_m = \\
& \varepsilon_{ikm}\{\mu_i\sigma_{ki}x_ix_me_i + \mu_i\sigma_{km}(1-x_i^2)e_m\}
\end{aligned}$$

Therefore,

$$A_\xi R(\xi, A_\xi e_i)e_k = \quad -\varepsilon_{ikm}\big\{\mu_i^2\sigma_{ki}x_ix_me_i\times\xi+\mu_i\mu_m\sigma_{km}(1-x_i^2)e_m\times\xi\big\} = \\ -\varepsilon_{ikm}\big\{\mu_i^2\sigma_{ki}x_ix_mN_i+\mu_i\mu_m\sigma_{km}(1-x_i^2)N_m\big\}$$

Thus,

$$2A_\xi Hm_\xi(e_i,e_k)=\varepsilon_{ikm}\big\{-\mu_i^2\sigma_{ki}x_ix_mN_i+\mu_k^2\sigma_{ki}x_kx_mN_k- \\ \big(\mu_i\mu_m\sigma_{km}(1-x_i^2)-\mu_k\mu_m\sigma_{im}(1-x_k^2)\big)N_m\big\}$$

So, finally

$$2\varepsilon_{ikm}TG_\xi(e_i,e_k)=x_ix_m(-\sigma_{ik}\mu_i^2+\mu_i\mu_k)N_i-x_kx_m(-\sigma_{ik}\mu_k^2+\mu_i\mu_k)N_k+ \\ \Big(\mu_i\mu_m-\mu_k\mu_m-\mu_i\mu_m\sigma_{km}(1-x_i^2)+\mu_k\mu_m\sigma_{im}(1-x_k^2)+ \\ \mu_i\mu_k(x_k^2-x_i^2)\Big)N_m= \\ x_ix_m\mu_i(-\sigma_{ik}\mu_i+\mu_k)N_i-x_kx_m\mu_k(-\sigma_{ik}\mu_k+\mu_i)N_k+ \\ \Big(\mu_i\mu_m(1-\sigma_{km})-\mu_k\mu_m(1-\sigma_{im})+\mu_i(\sigma_{km}\mu_m-\mu_k)x_i^2- \\ \mu_k(\sigma_{im}\mu_m-\mu_i)x_k^2\Big)N_m.$$

The proof is complete.

■

Theorem 2.1 *Let G be a three-dimensional unimodular Lie group with the left-invariant metric and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5). *Denote by ρ_1,ρ_2,ρ_3 the principal Ricci curvatures of the given group. Then the set of left-invariant totally geodesic unit vector fields can be described as follows.*

Table 1

ρ_1	ρ_2	ρ_3	μ_1	μ_2	μ_3	ξ
0	0	0	0	0	0	S
0	0	0	$\neq 0$	0	0	$\pm e_1,\ S \cap \{e_2, e_3\}_R$
0	0	0	0	$\neq 0$	0	$\pm e_2,\ S \cap \{e_1, e_3\}_R$
0	0	0	0	0	$\neq 0$	$\pm e_3,\ S \cap \{e_1, e_2\}_R$
2						$\pm e_1$
	2					$\pm e_2$
		2				$\pm e_3$
2	2					$S \cap \{e_1, e_2\}_R$
2		2				$S \cap \{e_1, e_3\}_R$
	2	2				$S \cap \{e_2, e_3\}_R$
2	2	2				S

where $S \cap \{e_i, e_k\}_R$ means the set of unit vectors in a plane, spanned by e_i and e_k, in the tangent space of the group at the unit element.

Proof. Rewrite the result of Lemma 2.2 for various combinations of indices.

$$(1,1) \quad x_1\mu_1\Big\{x_3(\sigma_{12}\mu_2 - \mu_1)N_2 - x_2(\sigma_{13}\mu_3 - \mu_1)N_3\Big\} = 0,$$

$$(2,2) \quad x_2\mu_2\Big\{x_3(\sigma_{21}\mu_1 - \mu_2)N_1 - x_1(\sigma_{23}\mu_3 - \mu_2)N_3\Big\} = 0,$$

$$(3,3) \quad x_3\mu_3\Big\{x_2(\sigma_{31}\mu_1 - \mu_3)N_1 - x_1(\sigma_{32}\mu_2 - \mu_3)N_2\Big\} = 0,$$

$$(1,2) \quad -x_1x_3\mu_1(\sigma_{12}\mu_1 - \mu_2)N_1 + x_2x_3\mu_2(\sigma_{12}\mu_2 - \mu_1)N_2 + \Big(\mu_1\mu_3(1-\sigma_{23}) - \mu_2\mu_3(1-\sigma_{13}) + \mu_1(\sigma_{23}\mu_3 - \mu_2)x_1^2 - \mu_2(\sigma_{13}\mu_3 - \mu_1)x_2^2\Big)N_3 = 0,$$

$$(2,3) \quad -x_2x_1\mu_2(\sigma_{23}\mu_2 - \mu_3)N_2 + x_3x_1\mu_3(\sigma_{23}\mu_3 - \mu_2)N_3 + \Big(\mu_2\mu_1(1-\sigma_{31}) - \mu_3\mu_1(1-\sigma_{21}) + \mu_2(\sigma_{31}\mu_1 - \mu_3)x_2^2 - \mu_3(\sigma_{21}\mu_1 - \mu_2)x_3^2\Big)N_1 = 0,$$

$$(3,1) \quad -x_3x_2\mu_3(\sigma_{13}\mu_3 - \mu_1)N_3 + x_1x_2\mu_1(\sigma_{13}\mu_1 - \mu_3)N_1 + \Big(\mu_3\mu_2(1-\sigma_{12}) - \mu_1\mu_2(1-\sigma_{32}) + \mu_3(\sigma_{12}\mu_2 - \mu_1)x_3^2 - \mu_1(\sigma_{32}\mu_2 - \mu_3)x_1^2\Big)N_2 = 0.$$

The vectors N_1, N_2 and N_3 are linearly dependent:

$$x_1N_1 + x_2N_2 + x_3N_3 = 0,$$

but linearly independent in pairs for general (not specific) field ξ.
The case $x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$.

The subcase 1: $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0$. All equations are fulfilled evidently. Therefore, *any left-invariant vector field is totally geodesic in this case*, and we get the first row in the Table 1.

The subcase 2: $\mu_1 = 0$, $\mu_2 \neq 0$ or $\mu_3 \neq 0$. Then from (2,2) and (3,3) we see, that $\mu_2 = 0, \mu_3 = 0$. Contradiction. In a similar way we exclude the cases when $\mu_i = 0$, but $\mu_k^2 + \mu_m^2 \neq 0$ for arbitrary triple of different indices (i, k, m).

The subcase 3: $\mu_1 \neq 0, \mu_2 \neq 0, \mu_3 \neq 0$. Since N_1, N_2 and N_3 are linearly independent in pairs, from (1,1), (2,2) and(3,3) we conclude:

$$\left\{\begin{array}{l} \sigma_{12}\mu_2 - \mu_1 = 0, \\ \sigma_{12}\mu_1 - \mu_2 = 0, \end{array}\right. , \quad \left\{\begin{array}{l} \sigma_{13}\mu_3 - \mu_1 = 0, \\ \sigma_{13}\mu_1 - \mu_3 = 0, \end{array}\right. \left\{\begin{array}{l} \sigma_{23}\mu_2 - \mu_3 = 0, \\ \sigma_{23}\mu_3 - \mu_2 = 0. \end{array}\right. \tag{13}$$

As a consequence,

$$\left\{\begin{array}{l} (\sigma_{12} - 1)(\mu_1 + \mu_2) = 0, \\ (\sigma_{13} - 1)(\mu_1 + \mu_3) = 0, \\ (\sigma_{23} - 1)(\mu_2 + \mu_3) = 0. \end{array}\right.$$

Taking into account (13), the rest of the equations yield

$$\left\{\begin{array}{l} \mu_1\mu_3(1 - \sigma_{23}) - \mu_2\mu_3(1 - \sigma_{13}) = 0, \\ \mu_1\mu_2(1 - \sigma_{13}) - \mu_1\mu_3(1 - \sigma_{12}) = 0, \\ \mu_2\mu_3(1 - \sigma_{12}) - \mu_1\mu_2(1 - \sigma_{23}) = 0. \end{array}\right.$$

Since $\mu_i \neq 0$ $(i = 1, 2, 3)$, we conclude $\sigma_{ik} = 1$ $(i, k = 1, 2, 3)$ and therefore $\rho_i = 2$ $(i = 1, 2, 3)$. This is the case of the last row in the Table 1.

The case $x_1 \neq 0, x_2 \neq 0, x_3 = 0$. In this case $x_1N_1 + x_2N_2 = 0$, but N_1, N_3 and N_2, N_3 are *linearly independent* in pairs. Rewrite the system for this case as follows.

$$\begin{array}{ll} (1,1) & \mu_1(\sigma_{13}\mu_3 - \mu_1) = 0, \\ (2,2) & \mu_2(\sigma_{23}\mu_3 - \mu_2) = 0, \\ (3,3) & \equiv 0 \end{array}$$

$$\begin{array}{ll} (1,2) & \mu_1\mu_3(1 - \sigma_{23}) - \mu_2\mu_3(1 - \sigma_{13}) + \mu_1(\sigma_{23}\mu_3 - \mu_2)x_1^2 - \\ & \hfill \mu_2(\sigma_{13}\mu_3 - \mu_1)x_2^2 = 0, \\ (2,3) & x_1^2\mu_2(\sigma_{23}\mu_2 - \mu_3) + \mu_1\mu_2(1 - \sigma_{31} - \mu_1\mu_3(1 - \sigma_{21}) + \\ & \hfill \mu_2(\sigma_{13}\mu_1 - \mu_3)x_2^2 = 0, \\ (3,1) & -x_2^2\mu_1(\sigma_{13}\mu_1 - \mu_3 + \mu_2\mu_3(1 - \sigma_{12}) - \mu_1\mu_2(1 - \sigma_{32}) - \\ & \hfill \mu_1(\sigma_{23}\mu_2 - \mu_3)x_1^2 = 0. \end{array}$$

Set $\mu_1 = \mu_2 = 0$. Then the system is fulfilled for arbitrary μ_3. The case $\mu_3 = 0$ is already considered. The case $\mu_3 \neq 0$ gives the $S \cap \{e_1, e_2\}_R$ it 3-rd row of the Table 1.
Set $\mu_1 = 0, \mu_2 \neq 0$. Then $\sigma_{12} = \mu_2\mu_3, \sigma_{13} = \mu_2\mu_3, \sigma_{23} = -\mu_2\mu_3$. The equation (2,2) yields $-\mu_2^2(\mu_3^2+1) = 0$. The contradiction.
Set $\mu_1 \neq 0, \mu_2 = 0$. Then $\sigma_{12} = \mu_1\mu_3, \sigma_{13} = -\mu_1\mu_3, \sigma_{23} = \mu_1\mu_3$. The equation (1,1) yields $-\mu_1^2(\mu_3^2+1) = 0$. The contradiction.
Set $\mu_1 \neq 0, \mu_2 \neq 0$. Then $\mu_1 = \sigma_{13}\mu_3, \mu_2 = \sigma_{23}\mu_3$ and the substitution into (1,2) yields

$$\mu_3^3(\mu_2 - \mu_1) = 0.$$

The case $\mu_3 = 0$ contradicts $\mu_1 \neq 0, \mu_2 \neq 0$, as one can see from (1,1) and (2,2). Thus, set $\mu_1 = \mu_2 = \mu \neq 0$. Then $\sigma_{13} = \sigma_{23} = \mu^2$ and from (1,1) and (2,2) we conclude

$$\mu\mu_3 - 1 = 0. \tag{14}$$

In this case we have

$$\sigma_{12} = 2 - \mu^2, \quad \sigma_{13} = \mu^2, \quad \sigma_{23} = \mu^2. \tag{15}$$

The substitution of (14) and (15) into the system yields the identity. Since $\mu\mu_3 = 1$ in our consideration means $\rho_1 = \rho_2 = 2$, we get the 8-th row of the Table 1.

The case $x_1 \neq 0, x_2 = 0, x_3 \neq 0$, after similar computations, resulting $S \cap \{e_1, e_3\}_R$ in the 3-rd row and the 9-th row of the Table 1.

The case $x_1 = 0, x_2 \neq 0, x_3 \neq 0$ resulting $S \cap \{e_2, e_3\}_R$ in the 4-rd row and the 10-th row of the Table 1.

The case $x_1 = 1, x_2 = 0, x_3 = 0$. In this case $N_1 = 0$ and the equations (1,1), (2,2), (3,3) and (2,3) are fulfilled regardless the geometry of the group. The equations (1,2) and (1,3) take the forms

$$(1,2) \quad \mu_1\mu_3(1-\sigma_{23}) - \mu_2\mu_3(1-\sigma_{13}) + \mu_1(\sigma_{23}\mu_3 - \mu_2) = 0$$
$$(1,3) \quad \mu_2\mu_3(1-\sigma_{12}) - \mu_1\mu_2(1-\sigma_{23}) - \mu_1(\sigma_{23}\mu_2 - \mu_3) = 0$$

After simplifications, we get

$$(1,2) \quad \sigma_{13}(\mu_2\mu_3 - 1) = 0,$$
$$(1,3) \quad \sigma_{12}(\mu_2\mu_3 - 1) = 0.$$

The case $\mu_2\mu_3 = 1$ means $\rho_1 = 2$ and we have the 5-th row of the Table 1. Consider the case $\sigma_{12} = 0, \sigma_{13} = 0$ which is equivalent to the system

$$\begin{cases} \mu_2\mu_3 = 0, \\ \mu_1(\mu_2 - \mu_3) = 0. \end{cases}$$

We have 4 possible solutions:

$$(i)\ \mu_1 = 0, \mu_2 = 0, \mu_3 = 0; \quad (ii)\ \mu_1 = 0, \mu_2 = 0, \mu_3 \neq 0;$$

$$(iii)\ \mu_1 = 0, \mu_2 \neq 0, \mu_3 = 0; \quad (iv)\ \mu_1 \neq 0, \mu_2 = 0, \mu_3 = 0.$$

The case (i) is already included into the 1-st row of the Table 1, the case (ii) is already included into $S \cap \{e_1, e_2\}_R$ case in the 4-st of the Table 1, the case (iii) is already included into $S \cap \{e_1, e_3\}_R$ case in the 3-rd row of the Table 1. The case (iv) is a new one and yields e_1 field in the 2-nd row of the Table 1.

The case $x_1 = 0, x_2 = 1, x_3 = 0$ yields e_2 into the 3-rd and 6-th rows of the Table 1.

The case $x_1 = 0, x_2 = 0, x_3 = 1$ yields e_3 into the 4-th and 7-th rows of the Table 1.

The proof is complete.

■

Now we specify the result of the Theorem 2.1 to each of the unimodular groups.

Theorem 2.2 *Let G be a three-dimensional unimodular Lie group with the left-invariant metric and let $\{e_i,\ i = 1, 2, 3\}$ be an orthonormal basis for the Lie algebra satisfying* (5). *Moreover, assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then the left-invariant unit vector fields of G are given as follows:*

G	*Conditions on* $\lambda_1, \lambda_2, \lambda_3$	*The sets of left-invariant totally geodesic unit vector fields*
SU(2)	$\lambda_1 = \lambda_2 = \lambda_3 = 2$	S
	$\lambda_1 = \lambda_2 = \lambda > \lambda_3 = 2$	$\pm e_3$
	$\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$	$S \cap \{e_1, e_2\}_R$
	$\lambda_1 = 2 > \lambda_2 = \lambda_3 = \lambda > 0$	$\pm e_1$
	$\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$	$S \cap \{e_2, e_3\}_R$
	$\lambda_1 > \lambda_2 > \lambda_3 > 0, \quad \lambda_m^2 - (\lambda_i - \lambda_k)^2 = 4$	$\pm e_m$ *(i,k,m=1,2,3)*
SL(2,R)	$\lambda_3^2 - (\lambda_1 - \lambda_2)^2 = 4$	$\pm e_3$
	$\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4$	$\pm e_1$
E(2)	$\lambda_1 = \lambda_2 > 0, \quad \lambda_3 = 0$	$\pm e_3$, $S \cap \{e_1, e_2\}_R$
	$\lambda_1^2 - \lambda_2^2 = 4,\ \lambda_1 > \lambda_2 > 0, \quad \lambda_3 = 0$	$\pm e_1$
E(1,1)	$\lambda_1^2 - \lambda_2^2 = -4,\ \lambda_1 > 0, \lambda_2 < 0, \quad \lambda_3 = 0$	$\pm e_2$
	$\lambda_1^2 - \lambda_2^2 = 4,\ \lambda_1 > 0, \lambda_2 < 0, \quad \lambda_3 = 0$	$\pm e_1$
Heisenberg group	$\lambda_1 = 2, \quad \lambda_2 = 0, \lambda_3 = 0$	$\pm e_1$
$R \oplus R \oplus R$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	S

where $S \cap \{e_i, e_k\}_R$ *means the set of unit vectors in a plane, spanned by* e_i *and* e_k*, in the tangent space of the group at the unit element.*

Proof.
The case SU(2). In this case $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. A simple calculation yields

$$\rho_m = 2\mu_i\mu_k = \frac{1}{2}(\lambda_m^2 - (\lambda_i - \lambda_k)^2).$$

Observe that $\rho_m = \rho_k$ if and only if $\lambda_m = \lambda_k$. From the Table 1 we now readout the cases

- if $\lambda_1 = \lambda_2 = \lambda_3 = 2$, then $\rho_1 = \rho_2 = \rho_3 = 2$ and each left-invariant unit vector field is totally geodesic one.

- if $\lambda_1 = \lambda_2 = \lambda > \lambda_3 = 2$, then $\rho_1 = \rho_2 = \frac{1}{2}(\lambda^2 - (\lambda-2)^2) = 2(\lambda-1) > 2$, $\rho_3 = 2$ and we have $\pm e_3$ as a unique totally geodesic left-invariant unit vector field.

- if $\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3$, then $\rho_1 = \rho_2 = \frac{1}{2}(\lambda^2 - (\lambda - \lambda_3)^2) = \frac{1}{2}(2\lambda\lambda_3 - \lambda_3^2)$, $\rho_3 = \frac{1}{2}\lambda_3^2 < 2$. Equalizing
$$\frac{1}{2}(2\lambda\lambda_3 - \lambda_3^2) = 2$$
we have $\lambda_3 = \lambda \pm \sqrt{\lambda^2 - 4}$. Since $\lambda_3 < \lambda$, the appropriate solution is $\lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$. In this case the set of totally geodesic left-invariant unit vector fields is $S \cap \{e_1, e_2\}_R$.

- if $\lambda_1 = 2 > \lambda_2 = \lambda_3 = \lambda > 0$, then $\rho_1 = 2$, $\rho_2 = \rho_3 = \frac{1}{2}(\lambda^2 - (\lambda-2)^2) = 2(\lambda - 1) < 2$ and we have a unique left-invariant totally geodesic unit vector field $\pm e_1$.

- if $\lambda_1 > 2 > \lambda_2 = \lambda_3 = \lambda > 0$, then $\rho_1 = \frac{1}{2}\lambda_1^2 > 2$ and $\rho_2 = \rho_3 = \frac{1}{2}(\lambda^2 - (\lambda - \lambda_1)^2) = \frac{1}{2}(2\lambda\lambda_1 - \lambda_1^2)$. Equalizing
$$\frac{1}{2}(2\lambda\lambda_1 - \lambda_1^2) = 2,$$
we find $\lambda_1 = \lambda \pm \sqrt{\lambda_2 - 4}$. Since $\lambda_1 > \lambda$, the appropriate solution is $\lambda_1 = \lambda + \sqrt{\lambda^2 - 4}$. In this case the set of totally geodesic left-invariant unit vector fields is $S \cap \{e_2, e_3\}_R$.

- if $\lambda_1 > \lambda_2 > \lambda_3 > 0$, then ρ_1, ρ_2 and ρ_3 are all different. In this case, if
$$\lambda_m^2 - (\lambda_i - \lambda_k)^2 = 4$$
for $m \neq i \neq k$, then the corresponding Ricci curvature $\rho_m = 2$ and we have $\pm e_m$ as a unique left-invariant totally geodesic unit vector field.

The case SL(2,R). In this case $\lambda_1 \geq \lambda_2 > 0$, $\lambda_3 < 0$ and the Ricci principal curvatures are
$$\rho_m = 2\mu_i\mu_k = \frac{1}{2}(\lambda_m^2 - (\lambda_i - \lambda_k)^2).$$

- if $\lambda_1 = \lambda_2 = \lambda > 0$, then $\rho_1 = \rho_2 = \frac{1}{2}(\lambda^2 - (\lambda - \lambda_3)^2) = \frac{1}{2}(2\lambda\lambda_3 - \lambda_3^2)$, $\rho_3 = \frac{1}{2}\lambda_3^2$. Equalizing
$$\frac{1}{2}(2\lambda\lambda_3 - \lambda_3^2) = 2$$
we have $\lambda_3 = \lambda \pm \sqrt{\lambda^2 - 4}$. Since $\lambda_3 < 0$, we have no appropriate solutions. Therefore, equalizing $\rho_3 = 2$, we have a unique case $\lambda_3 = -2$ and the vector field $\pm e_3$.

- if $\lambda_1 > \lambda_2 > 0$, then ρ_1, ρ_2 and ρ_3 are all different. In this case, consider separately the condition

  $$\lambda_m^2 - (\lambda_i - \lambda_k)^2 = 4$$

  for each m, i, k.

  For $m = 3$ we have

  $$\lambda_3^2 - (\lambda_1 - \lambda_2)^2 = 4.$$

  If λ_1, λ_2 and λ_3 satisfy this equation, then $\pm e_3$ is totally geodesic. Remark, that this case contains the case $\lambda_1 = \lambda_2$.

  For $m = 2$ we have

  $$\lambda_2^2 - (\lambda_1 - \lambda_3)^2 = 4.$$

  Since $\lambda_3 < 0$ we have $\lambda_1 - \lambda_3 > \lambda_1$. Therefore, $\lambda_2^2 - (\lambda_1 - \lambda_3)^2 < 0$. This contradiction shows that $\pm e_2$ is never totally geodesic.

  For $m = 1$ we have

  $$\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4.$$

  Since $\lambda_1 > \lambda_2$ we have $\pm e_1$ totally geodesic for all solutions of the equation above. Remark, that the solution necessarily satisfy $\lambda_1 - \lambda_2 > -\lambda_3$.

The case E(2). In this case $\lambda_1 \geq \lambda_2 > 0$, $\lambda_3 = 0$ and the Ricci principal curvatures are

$$\rho_1 = \frac{1}{2}(\lambda_1^2 - \lambda_2^2), \quad \rho_2 = -\rho_1 = \frac{1}{2}(\lambda_2^2 - \lambda_1^2), \quad \rho_3 = -\frac{1}{2}(\lambda_1 - \lambda_2)^2.$$

- if $\lambda_1 = \lambda_2 = \lambda > 0$ then $\rho_1 = \rho_2 = \rho_3 = 0$ and the group is flat. Make an auxiliary calculations:

  $$\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3) = 0, \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3) = 0,$$

  $$\mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3) = \lambda > 0.$$

  From the Table 1 we find $\pm e_3, S \cap \{e_1, e_2\}_R$.

- if $\lambda_1 > \lambda_2$, then we have one more condition $\rho_1 = 2$,i.e.

  $$\lambda_1^2 - \lambda_2^2 = 4$$

  which yields $\pm e_1$ as a totally geodesic field.

The case E(1,1). In this case $\lambda_1 > 0, \lambda_2 < 0$, $\lambda_3 = 0$ and the Ricci principal curvatures are

$$\rho_1 = \frac{1}{2}(\lambda_1^2 - \lambda_2^2), \quad \rho_2 = -\rho_1 = \frac{1}{2}(\lambda_2^2 - \lambda_1^2), \quad \rho_3 = -\frac{1}{2}(\lambda_1 - \lambda_2)^2.$$

In this case $\rho_3 < 0$, $\rho_1 \neq \rho_2$ and we have only two possible cases: either $\rho_1 = 2$ or $\rho_2 = 2$.

- $\rho_1 = 2$. In this case λ_1 and λ_2 should satisfy

$$\lambda_1^2 - \lambda_2^2 = 2,$$

which yields $\pm e_1$ as the totally geodesic field.

- $\rho_2 = 2$. In this case λ_1 and λ_2 should satisfy

$$\lambda_2^2 - \lambda_1^2 = 4$$

which yields $\pm e_2$ as a totally geodesic field.

The case of Heisenberg group. In this case $\lambda_1 > 0, \lambda_2 = \lambda_3 = 0$, and the Ricci principal curvatures are

$$\rho_1 = \frac{1}{2}\lambda_1^2, \quad \rho_2 = \rho_3 = -\frac{1}{2}\lambda_1^2.$$

In this case $\rho_2 < 0$, $\rho_3 < 0$ and we have only one possible case $\rho_1 = 2$ for $\lambda_1 = 2$, which yields $\pm e_1$ as the totally geodesic field.

The case $\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$. Here $\lambda_1 = \lambda_2 = \lambda_3$ and evidently all left-invariant unit vector fields are totally geodesic.

■

2.1 Geometrical characterization of totally geodesic unit vector fields

Let M be an odd-dimensional smooth manifold. Denote by ϕ, ξ, η a $(1,1)$ tensor field , a vector field and a 1-form on M respectively. A triple (ϕ, ξ, η) is called an *almost contact structure* on M if

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1 \tag{16}$$

for any vector field X on M. The manifold M with the almost contact structure is called an *almost contact* manifold.

If M is endowed with a Riemannian metric $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ such that

$$\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad \eta(X) = \langle \xi, X \rangle \tag{17}$$

for all vector fields X and Y on M, then a quadruple (ϕ, ξ, η, g) is called an *almost contact **metric** structure* and the manifold is called an *almost contact **metric** manifold.* The first of the conditions above is called a *compatibility condition* for ϕ and g.

If 2-form $d\eta$, given by

$$d\eta(X, Y) = \frac{1}{2}\big(X\eta(Y) - Y\eta(X) - \eta([X, Y])\big),$$

satisfies

$$d\eta(X,Y) = \langle X, \phi Y\rangle, \tag{18}$$

then the structure (ϕ,ξ,η,g) is called **contact metric structure** and the manifold with a contact metric structure is called by a **contact metric manifold**. A contact metric manifold is called *K-contact*, if ξ is a Killing vector field.

The Nijenhuis torsion of tensor field T of type $(1,1)$ is given by

$$[T,T](X,Y) = T^2[X,Y] + [TX,TY] - T[TX,Y] - T[X,TY]$$

and defines a $(1,2)$ tensor field on M. An almost contact structure (ϕ,ξ,η) is called *normal*, if

$$[\phi,\phi](X,Y) + 2d\eta(X,Y)\,\xi = 0. \tag{19}$$

Finally, a contact metric structure (ϕ,ξ,η,g) is called *Sasakian*, if it is normal. A manifold with Sasakian structure is called *Sasakian manifold*. In Sasakian manifold necessarily $\phi = A_\xi$ and $\eta = \langle \xi, \cdot\rangle$. The unit vector field ξ is called a *characteristic vector field* of the Sasakian structure and is a Killing one. This vector field is always totally geodesic [20].

In tree-dimensional case we have

Theorem 2.3 *[20] Let ξ be a unit Killing vector field on 3-dimensional Riemannian manifold M^3. If $\xi(M^3)$ is totally geodesic in T_1M^3 then either*

$$\big(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot\,\rangle\big)$$

is a Sasakian structure on M^3 or $M^3 = M^2 \times E^1$ metrically and ξ is the unit vector field of Euclidean factor.

Define the structure

$$\big(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot\,\rangle\big), \tag{20}$$

where the (1,1) tensor field is given by (10). Now we can give a geometrical description of totally geodesic unit vector fields.

Proposition 2.1 *Let ξ be a left invariant totally geodesic unit vector field on $SU(2)$ with the left invariant metric g and let $\{e_i,\ i = 1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5). *Assume in addition that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then*

$$\big(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot\,\rangle\big)$$

is the almost contact structure on $SU(2)$. Moreover,

- *if $\lambda_1 = \lambda_2 = \lambda_3 = 2$ or $\lambda_1 = \lambda_2 > \lambda_3 = 2$ or $\lambda_1 = 2 > \lambda_2 = \lambda_3$, then the structure is Sasakian;*

- *if* $\lambda_2 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$ *or* $\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$ *, then the structure is neither normal nor metric;*

- *if* $\lambda_1 > \lambda_2 > \lambda_3$*, then the structure is normal only for*

$$\xi = e_1, \quad \lambda_1 = \lambda_2 + \frac{1}{\lambda_2}, \quad \lambda_3 = \frac{1}{\lambda_2}, \quad \lambda_2 > 1$$

Proof. Consider the cases from Theorem 2.2.

• In the case of $\lambda_1 = \lambda_2 = \lambda_3 = 2$ we have $\mu_1 = \mu_2 = \mu_3 = 1$ and hence

$$A_\xi = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Therefore, the field ξ is the Killing one. By Theorem 2.3, the structure (20) is Sasakian.

In the case of $\lambda_1 = \lambda_2 = \lambda > \lambda_3 = 2$ we have $\mu_1 = 1, \mu_2 = 1, \mu_3 = \lambda - 1$ and $\xi = \pm e_3$. For $\xi = +e_3$ we find

$$A_\xi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and see that again ξ is the Killing unit vector field. Therefore, the structure (20) is Sasakian.

In the case of $\lambda_1 = 2 > \lambda_2 = \lambda_3 = \lambda > 0$ we have $\mu_1 = -1 + \lambda, \mu_2 = 1, \mu_3 = 1$ and $\xi = \pm e_1$. For $\xi = +e_1$ we find

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and see that again ξ is the Killing unit vector field. Therefore, the structure (20) is Sasakian.

• Consider the case $\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$ and $\xi = x_1 e_1 + x_2 e_2$. We have

$$\mu_1 = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4}), \quad \mu_2 = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4}), \quad \mu_3 = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}).$$

Set for brevity $\theta = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})$ and $\bar\theta = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})$. Then

$$\mu_1 = \theta, \quad \mu_2 = \theta, \quad \mu_3 = \bar\theta, \quad \theta\bar\theta = 1 \quad (\theta \neq 1, \bar\theta \neq 1)$$

and for this case we have

$$A_\xi = \begin{pmatrix} 0 & 0 & \bar\theta x_2 \\ 0 & 0 & -\bar\theta x_1 \\ -\theta x_2 & \theta x_1 & 0 \end{pmatrix}.$$

Since $\theta \neq \bar{\theta}$, the field ξ is never Killing one but geodesic. Indeed,

$$A_\xi \xi = \begin{pmatrix} 0 & 0 & \bar{\theta}x_2 \\ 0 & 0 & -\bar{\theta}x_1 \\ -\theta x_2 & \theta x_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \theta(-x_2 x_1 + x_1 x_2) \end{pmatrix} = 0.$$

The structure (20) is an almost contact one on $SU(2)$. Indeed,

$$\phi^2 = \begin{pmatrix} 0 & 0 & \bar{\theta}x_2 \\ 0 & 0 & -\bar{\theta}x_1 \\ -\theta x_2 & \theta x_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \bar{\theta}x_2 \\ 0 & 0 & -\bar{\theta}x_1 \\ -\theta x_2 & \theta x_1 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} -x_2^2 & x_1 x_2 & 0 \\ x_1 x_2 & -x_1^2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then

$$\phi^2 Z = \begin{pmatrix} -1 + x_1^2 & x_1 x_2 & 0 \\ x_1 x_2 & -1 + x_2^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = -Z + \langle \xi, Z \rangle \xi.$$

This structure is not metric one. For the compatibility condition (17) we have

$$\phi Z = \begin{pmatrix} \bar{\theta}x_2 z_3 \\ -\bar{\theta}x_1 z_3 \\ -\theta x_2 z_1 + \theta x_1 z_2 \end{pmatrix}, \quad \phi W = \begin{pmatrix} \bar{\theta}x_2 w_3 \\ -\bar{\theta}x_1 w_3 \\ -\theta x_2 w_1 + \theta x_1 w_2 \end{pmatrix}$$

and hence

$$\langle \phi Z, \phi W \rangle = \bar{\theta}^2 z_3 w_3 + \theta^2 (x_2^2 z_1 w_1 + x_1^2 z_2 w_2 - x_1 x_2 z_1 w_2 - x_1 x_2 z_2 w_1) =$$
$$\theta^2 (z_1 w_1 + z_2 w_2) + \bar{\theta}^2 x_3 w_3 - \langle \xi, Z \rangle \langle \xi, W \rangle \neq \langle Z, W \rangle - \langle \xi, Z \rangle \langle \xi, W \rangle.$$

This structure is not normal one. To prove this, check the normality condition (16). Find the Nijenhuis torsion of ϕ on e_1, e_2. We have

$$\phi e_1 = -\theta x_2 e_3, \quad \phi e_2 = \theta x_1 e_3, \quad \phi e_3 = \bar{\theta} x_2 e_1 - \bar{\theta} x_1 e_2,$$
$$[e_1, e_2] = 2\theta e_3, \quad [e_1, e_3] = -(\theta + \bar{\theta}) e_2, \quad [e_2, e_3] = (\theta + \bar{\theta}) e_1,$$
$$\phi^2 [e_1, e_2] = -2\theta e_3, \quad [\phi e_1, \phi e_2] = 0,$$
$$\phi[\phi e_1, e_2] = -\theta^2 x_2^2 (\theta + \bar{\theta}) e_3 = -\theta(\theta^2 + 1) x_2^2 e_3,$$
$$\phi[e_1, \phi e_2] = -\theta^2 (\theta + \bar{\theta}) x_1^2 e_3 = -\theta(\theta^2 + 1) x_1^2 e_3$$

and thus,

$$[\phi, \phi](e_1, e_2) = \theta(\theta^2 - 1) e_3 \neq 2d\eta(e_1, e_2)\xi.$$

In a similar way we can analyze the case $\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$ with the same result.

- Consider the case $\lambda_1 > \lambda_2 > \lambda_3$, $\xi = \pm e_i$. We have

$$\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3), \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3), \quad \mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)$$

Set $\xi = e_1$. The condition $\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4$ means that $\mu_2\mu_3 = 1$. The matrix A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu_3 \\ 0 & \mu_2 & 0 \end{pmatrix}.$$

Since $\mu_2 \neq \mu_3$, the field ξ is not a Killing one, but geodesic. The structure (20) is almost contact one. Indeed,

$$\phi^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mu_2\mu_3 & 0 \\ 0 & & -\mu_3\mu_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and hence

$$\phi^2 Z = -Z + \langle \xi, Z \rangle \xi.$$

The structure is *normal* if and only if

$$\lambda_1 = \lambda_2 + \frac{1}{\lambda_2}, \quad \lambda_3 = \frac{1}{\lambda_2}, \quad \lambda_2 > 1. \tag{21}$$

Indeed, remark that

$$\phi e_1 = 0, \quad \phi e_2 = \mu_2 e_3, \quad \phi e_3 = -\mu_3 e_2.$$

Now set $Z = e_1, W = e_2$. Then we have

$$\begin{aligned} &\phi^2[e_1, e_2] = \lambda_3 \phi^2 e_3 = -\lambda_3 e_3, \\ &[\phi e_1, \phi e_2] = 0, \\ &\phi[\phi e_1, e_2] = 0, \\ &\phi[e_1, \phi e_2] = \mu_2 \phi[e_1, e_3] = -\mu_2 \lambda_2 \phi e_2 = -\mu_2^2 \lambda_2 e_3, \\ &d\eta(e_1, e_2) = \tfrac{1}{2}(\langle e_1, \phi e_2 \rangle - \langle \phi e_1, e_2 \rangle) = 0. \end{aligned}$$

Therefore, the first necessary condition of normality is $\lambda_3 = \mu_2^2 \lambda_2$. If we remark that $\mu_2\mu_3 = 1$, we can rewrite this condition as

$$\lambda_3 \mu_3 = \lambda_2 \mu_2. \tag{22}$$

Set $Z = e_1, W = e_3$. Then we have

$$\begin{aligned}
&\phi^2[e_1, e_3] = -\lambda_2\phi^2 e_2 = \lambda_2 e_2,\\
&[\phi e_1, \phi e_3] = 0,\\
&\phi[\phi e_1, e_3] = 0,\\
&\phi[e_1, \phi e_3] = -\mu_3\phi[e_1, e_2] = -\mu_3\lambda_3\phi e_3 = \mu_3^2\lambda_3 e_2,\\
&d\eta(e_1, e_3) = \tfrac{1}{2}(\langle e_1, \phi e_3\rangle - \langle \phi e_1, e_3\rangle) = 0.
\end{aligned}$$

Therefore, the second necessary condition of normality is $\lambda_2 = \mu_3^2\lambda_3$. which is equivalent to (22).

Finally, set $Z = e_2, W = e_3$. Then we have

$$\begin{aligned}
&\phi^2[e_2, e_3] = \lambda_1\phi^2 e_1 = 0,\\
&[\phi e_2, \phi e_3] = -\mu_2\mu_3[e_3, e_2] = \lambda_1 e_1,\\
&\phi[\phi e_2, e_3] = 0,\\
&\phi[e_2, \phi e_3] = 0,\\
&d\eta(e_2, e_3) = \tfrac{1}{2}(\langle e_2, \phi e_3\rangle - \langle \phi e_2, e_3\rangle) = -\tfrac{1}{2}(\mu_3 + \mu_2) = -\tfrac{1}{2}\lambda_1.
\end{aligned}$$

These data satisfy (19). Expand the equation (22), namely

$$\lambda_3(\lambda_1 + \lambda_2 - \lambda_3) = \lambda_2(\lambda_1 - \lambda_2 + \lambda_3)$$

and perform rearrangements as follows:

$$\lambda_1(\lambda_3 - \lambda_2) + \lambda_3(\lambda_2 - \lambda_3) = \lambda_2(-\lambda_2 + \lambda_3).$$

Since $\lambda_2 \neq \lambda_3$, we get

$$\lambda_1 = \lambda_2 + \lambda_3.$$

Then

$$\mu_1 = 0, \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3) = \lambda_3, \quad \mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3) = \lambda_2$$

and, from the condition $\mu_2\mu_3 = 1$, we find

$$\lambda_2\lambda_3 = 1.$$

Since $\lambda_1 > \lambda_2 > \lambda_3$, we get (21).

The structure *is not metric*, since

$$\langle \phi Z, \phi W\rangle = \mu_3^2 z_3 w_3 + \mu_2^2 z_2 w_2 \neq \langle Z, W\rangle - \langle \xi, Z\rangle\langle \xi, W\rangle = z_2 w_2 + z_3 w_3.$$

Setting $\xi = e_2$, we get the normality condition of the form $\lambda_2 = \lambda_1 + \lambda_3$ which contradicts the condition $\lambda_1 > \lambda_2 > \lambda_3$. The structure is not metric.

Setting $\xi = e_3$, we get the normality condition of the form $\lambda_3 = \lambda_1 + \lambda_2$ which contradicts again the condition $\lambda_1 > \lambda_2 > \lambda_3$. The structure is not metric.

■

Proposition 2.2 *Let ξ be a left invariant totally geodesic unit vector field on $SL(2,R)$ with the left invariant metric g and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5)*. Assume in addition that $\lambda_1 \geq \lambda_2 > 0, \lambda_3 < 0$. Then*

$$\left(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot \rangle\right)$$

is the almost contact structure on $SL(2,R)$, where $\langle \cdot, \cdot \rangle$ is the scalar product with respect to g. Moreover, if

- $\lambda_1 = \lambda_2, \quad \lambda_3 = -2$*, then the structure is Sasakian;*
- $\lambda_3 = -\sqrt{4+(\lambda_1-\lambda_2)^2} < -2$ *or* $\lambda_1 = \sqrt{4+(\lambda_2-\lambda_3)^2}$ *, then the structure is neither normal nor metric.*

Proof. Consider the case of $\lambda_3 = -\sqrt{4+(\lambda_1-\lambda_2)^2} \leq -2$ and $\xi = e_3$. We have

$$\phi = A_\xi = \begin{pmatrix} 0 & -\mu_2 & 0 \\ \mu_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\mu_1\mu_2 = 1$.

If $\lambda_1 = \lambda_2$, then $\lambda_3 = -2$ and $\mu_1 = \mu_2 = 1$. Thus the field $\xi = e_3$ is the Killing one and the structure is Sasakian.

If $\lambda_1 > \lambda_2$, then $\lambda_3 < -2$. The structure is almost contact, since

$$\phi^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similar to the $SU(2)$ case, the structure is not metric and the normality condition for $\xi = e_3$ takes the form $\lambda_3 = \lambda_1 + \lambda_2$, which contradicts the sign conditions on λ_i.

Consider the case $\lambda_1 = \sqrt{4+(\lambda_2-\lambda_3)^2}$ and $\xi = e_1$. We have

$$\phi = A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu_3 \\ 0 & \mu_2 & 0 \end{pmatrix}$$

with $\mu_2\mu_3 = 1 \quad (\mu_2 \neq 1,\ \mu_3 \neq 1)$. Similar to $SU(2)$ case 3, the normality conditions take the form $\lambda_2 = \mu_3^2\lambda_3$ and $\lambda_3 = \mu_2^2\lambda_2$, that contradicts again the sign conditions on λ_i.

■

Proposition 2.3 *Let ξ be a left invariant totally geodesic unit vector field on $E(2)$ with the left invariant metric g and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5)*. Assume in addition that $\lambda_1 \geq \lambda_2 > 0, \lambda_3 = 0$.*

If $\lambda_1 = \lambda_2 = \lambda > 0$, then the group is flat. Moreover,

- *if $\xi = e_3$, then ξ is a parallel vector field on $E(2)$;*
- *if $\xi = x_1e_1 + x_2e_2$, then ξ moves along e_3 with a constant angle speed λ.*

If $\lambda_1 > \lambda_2 > 0$, then $\big(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot \rangle\big)$ is the almost contact structure on $E(2)$. This structure is neither metric nor normal.

Proof. Set $\lambda_1 = \lambda_2 = \lambda > 0$. Then $\mu_1 = 0, \mu_2 = 0, \mu_3 = \lambda$ and for $\xi = e_3$ we have $A_\xi = 0$. This means that ξ is a parallel vector field. If $\xi = x_1e_1 + x_2e_2$, then

$$A_\xi = \begin{pmatrix} 0 & 0 & \lambda x_2 \\ 0 & 0 & -\lambda x_1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $A_\xi^2 = 0$, the structure (20) is not almost contact one. The field ξ is not Killing but geodesic one. Moreover,

$$\nabla_{e_3}\xi = \lambda(x_1e_2 - x_2e_1).$$

This means that the field ξ moves along e_3-geodesics with a constant angle speed λ.

Set $\lambda_1 > \lambda_2 > 0$ and $\xi = e_1$. Then $\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2)$, $\mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2)$, $\mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2)$ and $\mu_2\mu_3 = 1$. We have

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu_3 \\ 0 & \mu_2 & 0 \end{pmatrix}$$

The structure (20) is an almost contact one. Similar to $SU(2)$ case 3 (with $\lambda_3 = 0$ setting), the normality condition of this structure is $\lambda_2 = \mu_3^2\lambda_3\ (= 0)$ which yields a contradiction.

■

Proposition 2.4 *Let ξ be a left invariant totally geodesic unit vector field on $E(1,1)$ with the left invariant metric and let $\{e_i,\ i = 1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5)*. Assume in addition that $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0$. Then*

$$\big(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot \rangle\big)$$

is the almost contact structure on $E(1,1)$. This structure is neither metric nor normal.

Proof. Consider the case $\lambda_1^2 - \lambda_2^2 = -4$, which is equivalent to $\mu_1\mu_2 = 1$, and $\xi = e_3$. Then

$$A_\xi = \begin{pmatrix} 0 & -\mu_2 & 0 \\ \mu_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the structure is almost contact one. As in previous cases, the structure is neither metric nor normal. A conclusion is true for the case of $\lambda_1^2-\lambda_2^2=4$ and $\xi=e_1$.

$\blacksquare$

Proposition 2.5 *Let ξ be a left invariant totally geodesic unit vector field on Heisenberg group with the left invariant metric and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (5)*. Moreover, assume that $\lambda_1>0, \lambda_2=0, \lambda_3=0$. Then*

$$\big(\phi=A_\xi,\ \xi,\ \eta=\langle\xi,\cdot\rangle\big)$$

is the Sasakian structure.

Proof. Indeed, for this case we have $\mu_1=-1, \mu_2=1, \mu_3=1$ and $\xi=e_1$. We have

$$A_\xi=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

which means that ξ is a Killing vector field and thus the structure is Sasakian.

$\blacksquare$

3 Non-unimodular case.

Choose the orthonormal frame e_1,e_2,e_3 as in (6). Then the Levi-Civita connection is given by the following table

∇	e_1	e_2	e_3
e_1	0	$\beta\, e_3$	$-\beta\, e_2$
e_2	$-\alpha\, e_2$	$\alpha\, e_1$	0
e_3	$-\delta\, e_3$	0	$\delta\, e_1$

(23)

For any left-invariant unit vector field $\xi=x_1e_1+x_2e_2+x_3e_3$ we have

$$\nabla_{e_1}\xi=\beta\ e_1\times\xi,\quad \nabla_{e_2}\xi=-\alpha\ e_3\times\xi,\quad \nabla_{e_3}\xi=\delta\ e_2\times\xi.$$

Set for brevity

$$e_1\times\xi=N_1,\quad e_3\times\xi=N_2,\quad e_2\times\xi=N_3,$$

or in explicit form

N_1	N_2	N_3
0	$-x_2$	x_3
$-x_3$	x_1	0
x_2	0	$-x_1$

. (24)

Then

$$A_\xi e_1 = -\beta\, N_1, \quad A_\xi e_2 = \alpha\, N_2, \quad A_\xi e_3 = -\delta\, N_3$$

and the matrix of A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & -\alpha\, x_2 & -\delta\, x_3 \\ \beta\, x_3 & \alpha\, x_1 & 0 \\ -\beta\, x_2 & 0 & \delta\, x_1 \end{pmatrix} \tag{25}$$

A direct computation gives the following result.

Lemma 3.1 *The derivatives $(\nabla_{e_i} A_\xi)e_k$ of the Weingarten operator A_ξ for the left invariant unit vector field are as in the following table.*

	e_1	e_2	e_3
e_1	$-\beta^2(x_1e_1 - \xi)$	$\beta\,\delta\, N_3 + \beta\,\alpha\, x_1 e_3$	$\beta\,\alpha\, N_2 - \beta\,\delta\, x_1 e_2$
e_2	$\alpha^2 N_2 + \beta\,\alpha\, x_3 e_1$	$\beta\,\alpha\, N_1 - \alpha^2(x_3e_3 - \xi)$	$\alpha\,\delta\, x_3 e_2$
e_3	$-\delta^2 N_3 - \beta\,\delta\, x_2 e_1$	$\alpha\,\delta\, x_2 e_3$	$\beta\,\delta\, N_1 - \delta^2(x_2e_2 - \xi)$

Proof. By definition,

$$(\nabla_{e_i} A_\xi)e_k = \nabla_{\nabla_{e_i} e_k}\xi - \nabla_{e_i}\nabla_{e_k}\xi.$$

Using the Table (23), we can easily fill out the table

$\nabla_{\nabla_{e_i} e_k}\xi$	e_1	e_2	e_3
e_1	0	$\beta\,\delta\, N_3$	$\beta\,\alpha\, N_2$
e_2	$\alpha^2 N_2$	$\beta\,\alpha\, N_1$	0
e_3	$-\delta^2 N_3$	0	$\beta\,\delta\, N_1$

and the table

∇	ξ	$\nabla_{e_1}\xi$	$\nabla_{e_2}\xi$	$\nabla_{e_3}\xi$
e_1	$\beta\, N_1$	$\beta^2(x_1e_1 - \xi)$	$-\beta\,\alpha\, x_1 e_3$	$\beta\,\delta\, x_1 e_2$
e_2	$-\alpha\, N_2$	$-\beta\,\alpha\, x_3 e_1$	$\alpha^2(x_2e_2 - \xi)$	$-\alpha\,\delta\, x_3 e_2$
e_3	$\delta\, N_3$	$\beta\,\delta\, x_2 e_1$	$-\alpha\,\delta\, x_2 e_3$	$\delta^2(x_3e_3 - \xi)$

Now, the result follows immediately.

■

By the straightforward application of Codazzi equation and Lemma 3.1 we can easily prove the following.

Lemma 3.2 *The curvature operator of the non-unimodular group with respect to the chosen frame takes the form*

$$
\begin{aligned}
R(e_1,e_2)\xi &= \alpha^2 N_2 + \beta(\alpha-\delta)N_3,\\
R(e_1,e_3)\xi &= -\delta^2 N_3 - \beta(\alpha-\delta)N_2\\
R(e_2,e_3)\xi &= \alpha\,\delta\, N_1
\end{aligned}
$$

Now, everything is prepared for the calculation of the components of total geodesity form (4).

Lemma 3.3 *Let G be non-unimodular Lie group with the basis, satisfying* (6)*. Then the left-invariant unit vector field $\xi = x_1e_1+x_2e_2+x_3e_3$ is totally geodesic if and only if it satisfies the following equations:*

$$
(1,1)\quad \beta\,x_1\Big\{\big[\beta\,[1+\alpha\,(\alpha-\delta)]x_2+\alpha^3x_3\big]N_2 - \big[\beta\,[1-\delta\,(\alpha-\delta)]x_3-\delta^3x_2\big]N_3\Big\}=0,
$$

$$
(2,2)\quad \alpha\left\{\Big[\beta\,[1+\alpha^2(1-x_3^2)]-[\alpha+\beta^2(\alpha-\delta)]x_2x_3\Big]N_1+ \alpha\Big[1+\delta^2\Big]x_1x_3N_3\right\}=0,
$$

$$
(3,3)\quad \delta\left\{\Big[\beta\,[1+\delta^2(1-x_2^2)]+[\delta-\beta^2(\alpha-\delta)]x_2x_3\Big]N_1- \delta\Big[1+\alpha^2\Big]x_1x_2N_2\right\}=0,
$$

$$
\begin{aligned}
(1,2)\quad &\beta\,x_1\Big[[\alpha+\beta^2(\alpha-\delta)]x_2+\beta\,\alpha^2x_3\Big]N_1+\\
&\alpha\Big[\alpha\,[1+\alpha^2(1-x_3^2)]-\beta\,[1+\alpha\,(\alpha-\delta)]x_2x_3\Big]N_2+\\
&\Big[\alpha\,\delta\,[\beta\,\delta\,(1-x_1^2)-\delta^2x_2x_3+\beta\,(\alpha-\delta)(1-x_3^2)]+\beta\,\alpha\,(x_3^2-x_1^2)+\beta\,\delta\Big]N_3=0,
\end{aligned}
$$

$$
\begin{aligned}
(1,3)\quad &\beta\,x_1\Big[[\delta-\beta^2(\alpha-\delta)]x_3-\beta\,\delta^2x_2\Big]N_1-\\
&\Big[\alpha\,\delta\,[\alpha\,\beta\,(1-x_1^2)+\alpha^2x_2x_3-\beta\,(\alpha-\delta)(1-x_2^2)]+\beta\,\alpha+\beta\,\delta\,(x_2^2-x_1^2)\Big]N_2+\\
&\delta\Big[\beta\,[-1+\delta\,(\alpha-\delta)]x_2x_3-\delta\,[1+\delta^2(1-x_2^2)]\Big]N_3=0,
\end{aligned}
$$

$$
\begin{aligned}
(2,3)\quad &\Big[\beta\,[\alpha\,\delta\,(\alpha+\delta)x_2x_3-\beta\,(\alpha-\delta)(\alpha\,(1-x_3^2)+\delta\,(1-x_2^2))]+\alpha\,\delta\,(x_2^2-x_3^2)\Big]N_1+\\
&\alpha\,\delta\Big[1+\alpha^2\Big]x_1x_3N_2-\alpha\,\delta\Big[1+\delta^2\Big]x_1x_2N_3=0.
\end{aligned}
$$

The proof consists of rather long calculations of the corresponding components $TG_\xi(e_i, e_k)$ for various combinations of (i,k), similar to the calculations in the unimodular case.

The analysis of the Lemma 3.3 we split into two subcases.

Theorem 3.1 *Let G be non-unimodular Lie group with the basis* (6)*. Let ξ be a left invariant unit vector field which does not belong to the unimodular kernel of the Lie algebra at the origin. Then ξ is never totally geodesic.*

Proof. By the hypothesis, $x_1 \neq 0$. From (24) it follows that $N_2 \neq 0$, $N_3 \neq 0$ and they are always linearly independent. Moreover, the vectors N_1 and N_3 are linearly dependent if and only if $x_3 = 0$. If $x_3 \neq 0$, then the equation $(2,2)$ implies $x_3 = 0$ and we come to a contradiction.

Set $x_3 = 0$. If $x_2 \neq 0$, then N_1 and N_2 are linearly independent and $(3,3)$ implies $\delta = 0$. In this case we can rewrite $(1,1)$ as $\beta^2 x_1 x_2 (1+\alpha^2) N_2 = 0$ and we have $\beta = 0$. In this case the equation $(1,2)$ takes the form $\alpha^2(1+\alpha^2)N_2 = 0$ and we have a contradiction.

Set $x_3 = x_2 = 0$. In this case $\xi = e_1$, $N_1 = 0$, $N_2 = e_2$ and $N_3 = -e_3$. The equation $(1,2)$ takes the form $\alpha^2(1+\alpha^2)N_2 = 0$. Contradiction.

■

Theorem 3.2 *Let G be non-unimodular Lie group with the basis as above. Let ξ be a left invariant totally geodesic unit vector field from the unimodular kernel of the Lie algebra at the origin. Then either*

$$\beta = \delta = 0 \quad \textit{and} \quad \xi = \pm e_3$$

or

$$\beta = \theta = \pm 1, \quad \alpha\delta = -1 \quad \textit{and} \quad \pm\,\xi = \theta \frac{1}{\sqrt{1+\alpha^2}}\, e_2 + \frac{\alpha}{\sqrt{1+\alpha^2}}\, e_3,$$

Proof. Suppose $\xi = x_2 e_2 + x_3 e_3$. Since $x_1 = 0$, we have $N_1 \neq 0$ and N_1 is linearly independent with either N_2 or N_3.

Suppose $\beta = 0$. Then $(2,2)$ implies $-\alpha^2 x_2 x_3 = 0$ and we have the following cases.

- Case $x_3 = 0$. Then $N_1 = \pm e_3$, $N_2 = \mp e_1$, $N_3 = 0$ and the equation $(1,2)$ takes the form $\alpha^2(1+\alpha^2)N_2 = 0$. Contradiction.

- Case $x_2 = 0$. Then $N_1 = \mp e_2$, $N_2 = 0$, $N_3 = \pm e_1$. The equation $(1,3)$ then takes the form $-\delta^2(1+\delta^2)N_3 = 0$ and we should set $\delta = 0$. It is easy to check that if $\beta = \delta = 0$, then all equations are fulfilled. Moreover, the field $\xi = \pm e_3$ becomes a parallel vector field, since $\nabla\xi = 0$.

Suppose $\beta \neq 0$, $\delta = 0$. Then $(1,3)$ implies $\beta\,\alpha\,N_2 = 0$ and we have $x_2 = 0$. In this case $x_3^2 = 1$ and $(2,2)$ yields $\alpha\,\beta\,N_1 = 0$. Contradiction.

Suppose $\beta \neq 0, \delta \neq 0$. In this case a direct analysis of the system becomes too complicated. Fortunately, we can apply to this case a different method based on the explicit expression for the second fundamental form of $\xi(M^n) \subset T_1M^n$ [18].

Let ξ be a unit vector field on a Riemannian manifold M^{n+1}. The components of second fundamental form of $\xi(M) \subset T_1M^{n+1}$ can be given by

$$\begin{aligned}\tilde\Omega_{\sigma|ij} = \ & \tfrac12\Lambda_{\sigma ij}\Big\{-\big\langle(\nabla_{e_i}A_\xi)e_j + (\nabla_{e_j}A_\xi)e_i, f_\sigma\big\rangle + \\ & \qquad\qquad \lambda_\sigma\left[\lambda_j\big\langle R(e_\sigma,e_i)\xi, f_j\big\rangle + \lambda_i\big\langle R(e_\sigma,e_j)\xi, f_i\big\rangle\right]\Big\},\end{aligned}$$

where $\Lambda_{\sigma ij} = [(1+\lambda_\sigma^2)(1+\lambda_i^2)(1+\lambda_j^2)]^{-1/2}$, $\lambda_0 = 0, \lambda_1, \dots, \lambda_n$ are the singular values of the matrix A_ξ and $e_0, e_1, \dots, e_n; f_1, \dots, f_n$ are the orthonormal frames of singular vectors ($i, j = 0, 1, \dots, n$; $\sigma = 1, \dots, n$).

Since $x_1 = 0$, the matrix (25) takes the form

$$A_\xi = \begin{pmatrix} 0 & -\alpha\,x_2 & -\delta\,x_3 \\ \beta\,x_3 & 0 & 0 \\ -\beta\,x_2 & 0 & 0 \end{pmatrix}$$

Denote by $\tilde e_0$, $\tilde e_1$, $\tilde e_2$; $\tilde f_1$, $\tilde f_2$ the orthonormal singular frames of A_ξ. The matrix $A_\xi^tA_\xi$ takes the form

$$A_\xi^tA_\xi = \begin{pmatrix} \beta^2 & 0 & 0 \\ 0 & \alpha^2x_2^2 & \alpha\,\delta\,x_2x_3 \\ 0 & \alpha\,\delta\,x_2x_3 & \delta^2x_3^2 \end{pmatrix}. \tag{26}$$

The eigenvalues are $\left[0, \beta^2, \alpha^2x_2^2 + \delta^2x_3^2\right]$. Denote $m = \sqrt{\alpha^2x_2^2 + \delta^2x_3^2}$. Then the singular values are

$$\lambda_0 = 0,\ \lambda_1 = |\beta\,|,\ \lambda_2 = m.$$

The singular frame $\tilde e_0$, $\tilde e_1$, $\tilde e_2$ consists of the eigenvectors of the matrix (26), namely

$$\tilde e_0 = \frac1m\big(-\delta\,x_3\,e_2 + \alpha\,x_2\,e_3\big), \quad \tilde e_1 = e_1, \quad \tilde e_2 = \frac1m\big(\alpha\,x_2\,e_2 + \delta\,x_3\,e_3\big)$$

To find $\tilde f_1$ and $\tilde f_2$, calculate $A_\xi\,\tilde e_1$ and $A_\xi\,\tilde e_2$:

$$A_\xi\,\tilde e_1 = \beta\,\big(x_3\,e_2 - x_2\,e_3\big), \quad A_\xi\,\tilde e_2 = -m\,e_1.$$

Denote $\varepsilon = sign(\beta)$. Then

$$\tilde{f}_1 = \varepsilon\big(x_3\, e_2 - x_2\, e_3\big), \qquad \tilde{f}_2 = -e_1.$$

Now we have

$$\tilde{\Omega}_{\sigma|00} = -\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\langle(\nabla_{\tilde{e}_0}A_\xi)\,\tilde{e}_0,\ \tilde{f}_\sigma\big\rangle.$$

If ξ is totally geodesic, then ξ satisfy

$$0 = (\nabla_{\tilde{e}_0}A_\xi)\,\tilde{e}_0 = \nabla_{\tilde{e}_0}(A_\xi\,\tilde{e}_0) - A_\xi\nabla_{\tilde{e}_0}\,\tilde{e}_0 = A_\xi A_{\tilde{e}_0}\,\tilde{e}_0$$

Since (25) is applicable to any left-invariant unit vector field, we easily calculate

$$A_{\tilde{e}_0}\,\tilde{e}_0 = \frac{1}{m^2}\begin{pmatrix} 0 & \alpha\,\delta\,x_3 & -\delta\,\alpha\,x_2 \\ \beta\,\alpha\,x_2 & 0 & 0 \\ \beta\,\delta\,x_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\delta\,x_3 \\ \alpha\,x_2 \end{pmatrix} =$$

$$-\frac{1}{m^2}\alpha\,\delta\,(\delta\,x_3^2+\alpha\,x_2^2)\ \tilde{e}_1.$$

Therefore,

$$A_\xi A_{\tilde{e}_0}\,\tilde{e}_0 = -\frac{1}{m^2}\alpha\,\delta\,(\delta\,x_3^2+\alpha\,x_2^2)A_\xi\,\tilde{e}_1 = -\varepsilon\beta\,\alpha\,\delta\,(\delta\,x_3^2+\alpha\,x_2^2)\,\tilde{f}_1.$$

Since $\beta \neq 0, \alpha \neq 0$ and $\delta \neq 0$, we have

$$\begin{cases} \alpha\,x_2^2+\delta\,x_3^2 = 0, \\ x_2^2+x_3^2 = 1. \end{cases}$$

Solving the system, we get

$$x_2^2 = \frac{-\delta}{\alpha-\delta}, \quad x_3^2 = \frac{\alpha}{\alpha-\delta}.$$

Remind that $\alpha+\delta > 0, \quad \alpha \geq \delta$ by the choice of the frame. Therefore, the solution exists, if $\delta < 0$ and, as a consequence, $\alpha > 0$. Thus,

$$\xi = \pm\sqrt{\frac{-\delta}{\alpha-\delta}}\ e_2 \pm \sqrt{\frac{\alpha}{\alpha-\delta}}\ e_3.$$

Denote $\theta = \pm 1$. Without loss of generality we can set

$$\xi = \theta\sqrt{\frac{-\delta}{\alpha-\delta}}\ e_2 + \sqrt{\frac{\alpha}{\alpha-\delta}}\ e_3.$$

As a consequence

$$m = \sqrt{\alpha^{2}\frac{-\delta}{\alpha-\delta} + \delta^{2}\frac{\alpha}{\alpha-\delta}} = \sqrt{-\alpha\,\delta}.$$

Moreover

$$\begin{aligned} \frac{\alpha}{m}x_2 &= \theta\,\frac{\alpha}{\sqrt{-\alpha\,\delta}}\sqrt{\frac{-\delta}{\alpha-\delta}} = \theta\,x_3, \\ \frac{\delta}{m}x_3 &= \frac{\delta}{\sqrt{-\alpha\,\delta}}\sqrt{\frac{\alpha}{\alpha-\delta}} = \frac{-\sqrt{(-\delta\,)^2}}{\sqrt{-\alpha\,\delta}}\sqrt{\frac{\alpha}{\alpha-\delta}} = -\theta\,x_2 \end{aligned}$$

and we have

$$\begin{aligned} \tilde e_0 &= \tfrac{1}{m}\big(-\delta\,x_3\,e_2 + \alpha\,x_2\,e_3\big) = \theta\,\xi, \\ \tilde e_1 &= e_1 = -\,\tilde f_2, \\ \tilde e_2 &= \tfrac{1}{m}\big(\alpha\,x_2\,e_2 + \delta\,x_3\,e_3\big) = \theta(x_3\,e_2 - x_2\,e_3) = \theta\varepsilon\ \tilde f_1. \end{aligned}$$

With respect to this frame, we have

$$\begin{aligned} A_\xi\,\tilde e_0 &= A_\xi \xi = 0, \\ A_\xi\,\tilde e_1 &= |\beta\,|\,\tilde f_1 = \theta\varepsilon|\beta\,|\ \tilde e_2 = \theta\beta\ \tilde e_2 \\ A_\xi\,\tilde e_2 &= m\,\tilde f_2 = -m\ \tilde e_1 \end{aligned}$$

and the matrix A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -m \\ 0 & \theta\beta & 0 \end{pmatrix}.$$

A simple calculation yields

∇	$\tilde e_0$	$\tilde e_1$	$\tilde e_2$
$\tilde e_0$	0	$-\theta m\,\tilde e_2$	$\theta m\,\tilde e_1$
$\tilde e_1$	$-\beta\ \tilde e_2$	0	$\beta\ \tilde e_0$
$\tilde e_2$	$\theta m\,\tilde e_1$	$-\theta m\,\tilde e_0 - (\alpha+\delta\,)\,\tilde e_2$	$(\alpha+\delta\,)\,\tilde e_1$

. (27)

With respect to new frame, the derivatives $(\nabla_{\tilde e_i} A_\xi)\,\tilde e_k$ form the following Table.

	$\tilde e_0$	$\tilde e_1$	$\tilde e_2$
$\tilde e_0$	0	$-m(\theta m - \beta\,)\ \tilde e_1$	$m(\theta m - \beta\,)\ \tilde e_2$
$\tilde e_1$	$-m\beta\ \tilde e_1$	$\theta\beta^{2}\ \tilde e_0$	0
$\tilde e_2$	$-m\beta\ \tilde e_2$	$-(\alpha+\delta\,)(m-\theta\beta\,)\ \tilde e_1$	$\theta m^2\,\tilde e_0 + (\alpha+\delta\,)(m-\theta\beta\,)\ \tilde e_2$

.

Finally, the necessary components of the curvature operator can be found from the latter Table and take the form

$$
\begin{aligned}
R(\tilde e_0,\, \tilde e_1)\xi &= m(\theta m - 2\beta\,)\ \tilde e_1,\\
R(\tilde e_0,\, \tilde e_2)\xi &= -\theta m^2\ \tilde e_2,\\
R(\tilde e_1,\, \tilde e_2)\xi &= -(\alpha\,+\delta\,)(m-\theta\beta\,)\ \tilde e_1.
\end{aligned}
\tag{28}
$$

Remark, also, that

$$\tilde f_1 = \theta\varepsilon\,\tilde e_2, \qquad \tilde f_2 = -\,\tilde e_1.$$

Now, we can find all the entries of the matrices $\tilde\Omega_\sigma$.

$$\tilde\Omega_{1|10} = \frac{-\big\langle(\nabla_{\tilde e_1}A_\xi)\,\tilde e_0 + (\nabla_{\tilde e_0}A_\xi)\,\tilde e_1,\ \tilde f_1\big\rangle + \lambda_1^2\big\langle R(\tilde e_1,\, \tilde e_0)\xi,\ \tilde f_1\big\rangle}{2(1+\lambda_1^2)} =$$

$$\frac{-\big\langle -\theta m^2\,\tilde e_1, \theta\varepsilon\,\tilde e_2\big\rangle + \beta^{\,2}\big\langle -m(\theta m-2\beta\,)\ \tilde e_1, \theta\varepsilon\,\tilde e_2\big\rangle}{2(1+\beta^{\,2})} = 0.$$

$$\tilde\Omega_{1|20} = \frac{-\big\langle(\nabla_{\tilde e_2}A_\xi)\,\tilde e_0 + (\nabla_{\tilde e_0}A_\xi)\,\tilde e_2,\ \tilde f_1\big\rangle + \lambda_1\lambda_2\big\langle R(\tilde e_1,\, \tilde e_0)\xi,\ \tilde f_2\big\rangle}{2\sqrt{(1-\lambda_1^2)(1+\lambda_2^2)}} =$$

$$\frac{-\big\langle m(\theta m-2\beta\,)\,\tilde e_2, \theta\varepsilon\,\tilde e_2\big\rangle + |\beta\,|m\big\langle -m(\theta m-2\beta\,)\,\tilde e_1, -\tilde e_1\big\rangle}{2\sqrt{(1+\beta^{\,2})(1+m^2)}} =$$

$$\frac{m(m-2\theta\beta\,)(m|\beta\,|-\theta\varepsilon)}{2\sqrt{(1-\beta^{\,2})(1-m^2)}} = \frac{\varepsilon m(\theta m-2\beta\,)(m\beta\,-\theta)}{2\sqrt{(1+\beta^{\,2})(1+m^2)}} =$$

$$\frac{\varepsilon\theta m(m-2\theta\beta\,)(m\beta\,-\theta)}{2\sqrt{(1+\beta^{\,2})(1+m^2)}}.$$

$$\tilde\Omega_{1|11} = \frac{-\big\langle(\nabla_{\tilde e_1}A_\xi)\,\tilde e_1,\ \tilde f_1\big\rangle}{\sqrt{(1+\lambda_1^2)^3}} = \frac{-\big\langle\theta\beta^{\,2}\,\tilde e_0, \theta\varepsilon\,\tilde e_1\big\rangle}{(1+\beta^{\,2})\sqrt{(1+\beta^{\,2})}} = 0.$$

$$\tilde\Omega_{1|12} = \frac{-\big\langle(\nabla_{\tilde e_1}A_\xi)\,\tilde e_2 + (\nabla_{\tilde e_2}A_\xi)\,\tilde e_1,\ \tilde f_1\big\rangle + \lambda_1^2\big\langle R(\tilde e_1,\, \tilde e_2)\xi,\ \tilde f_1\big\rangle}{2\sqrt{(1+\lambda_1^2)^2(1+\lambda_2^2)}} =$$

$$\frac{\big\langle(\alpha\,+\delta\,)(m-\theta\beta\,)\ \tilde e_1, \theta\varepsilon\,\tilde e_2\big\rangle + \beta^{\,2}\big\langle -(\alpha\,+\delta\,)(m-\theta\beta\,)\ \tilde e_1, \theta\varepsilon\,\tilde e_2\big\rangle}{2(1+\beta^{\,2})\sqrt{1+m^2}} = 0.$$

$$\tilde{\Omega}_{1|22} = \frac{-\big\langle (\nabla_{\tilde{e}_2} A_\xi)\, \tilde{e}_2,\, \tilde{f}_1 \big\rangle + \lambda_1\lambda_2 \big\langle R(\tilde{e}_1,\, \tilde{e}_2)\xi,\, \tilde{f}_2 \big\rangle}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)^2}} =$$

$$\frac{-\big\langle \theta m^2\, \tilde{e}_0 + (\alpha+\delta)(m-\theta\beta)\ \tilde{e}_2, \theta\varepsilon\, \tilde{e}_2 \big\rangle + |\beta|m\big\langle (\alpha+\delta)(m-\theta\beta)\ \tilde{e}_1,\, \tilde{e}_1 \big\rangle}{\sqrt{(1+\beta^2)}\,(1+m^2)} =$$

$$\frac{-\theta\varepsilon(\alpha+\delta)(m-\theta\beta) + |\beta|m(\alpha+\delta)(m-\theta\beta)}{\sqrt{(1+\beta^2)}\,(1+m^2)} =$$

$$\frac{\varepsilon\theta(\alpha+\delta)(\theta m-\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)}\,(1+m^2)}.$$

Summarizing, we get

$$\tilde{\Omega}_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\varepsilon\theta m(m-2\theta\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)(1+m^2)}} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\varepsilon\theta m(m-2\theta\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)(1+m^2)}} & 0 & \frac{\varepsilon\theta(\alpha+\delta)(\theta m-\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)(1+m^2)}} \end{pmatrix}.$$

In a similar way, we find

$$\tilde{\Omega}_{2|10} = \frac{-\big\langle (\nabla_{\tilde{e}_1} A_\xi)\, \tilde{e}_0 + (\nabla_{\tilde{e}_0} A_\xi)\, \tilde{e}_1,\, \tilde{f}_2 \big\rangle + \lambda_2\lambda_1 \big\langle R(\tilde{e}_2,\, \tilde{e}_0)\xi,\, \tilde{f}_1 \big\rangle}{2\sqrt{(1+\lambda_2^2)(1+\lambda_1^2)}} =$$

$$\frac{-\big\langle -\theta m^2\, \tilde{e}_1, -\tilde{e}_1 \big\rangle + m\,|\beta|\big\langle \theta m^2\, \tilde{e}_2, \theta\varepsilon\, \tilde{e}_2 \big\rangle}{2\sqrt{(1+m^2)(1+\beta^2)}} = \frac{m^2(m\beta-\theta)}{2\sqrt{(1+m^2)(1+\beta^2)}}.$$

$$\tilde{\Omega}_{2|20} = \frac{-\big\langle (\nabla_{\tilde{e}_2} A_\xi)\, \tilde{e}_0 + (\nabla_{\tilde{e}_0} A_\xi)\, \tilde{e}_2,\, \tilde{f}_2 \big\rangle}{2\sqrt{(1+\lambda_2^2)^2}} = \frac{-\big\langle (m(\theta m-\beta)\, \tilde{e}_2, -\tilde{e}_1 \big\rangle}{2(1+m^2)} = 0.$$

$$\tilde{\Omega}_{2|11} = \frac{-\big\langle (\nabla_{\tilde{e}_1} A_\xi)\, \tilde{e}_1,\, \tilde{f}_2 \big\rangle + \lambda_2\lambda_1 \big\langle R(\tilde{e}_2,\, \tilde{e}_1)\xi,\, \tilde{f}_1 \big\rangle}{\sqrt{(1+\lambda_2^2)(1+\lambda_1^2)}} =$$

$$\frac{-\big\langle \theta\beta^2\, \tilde{e}_0,\, \tilde{e}_1 \big\rangle + m|\beta|\big\langle (\alpha+\delta)(\theta m-\beta)\ \tilde{e}_1, \theta\varepsilon\, \tilde{e}_2 \big\rangle}{\sqrt{(1+m^2)(1+\beta^2)}} = 0.$$

$$\tilde{\Omega}_{2|12} = \frac{-\big\langle (\nabla_{\tilde e_1} A_\xi)\,\tilde e_2 + (\nabla_{\tilde e_2} A_\xi)\,\tilde e_1,\ \tilde f_2 \big\rangle + \lambda_2^2 \big\langle R(\tilde e_2,\ \tilde e_1)\xi,\ \tilde f_2 \big\rangle}{2\sqrt{(1+\lambda_2^2)^2(1+\lambda_1^2)}} =$$

$$\frac{\big\langle (\alpha+\delta)(m-\theta\beta)\ \tilde e_1, -\tilde e_1\big\rangle + m^2\big\langle (\alpha+\delta)(m-\theta\beta)\ \tilde e_1, -\tilde e_1\big\rangle}{2(1+m^2)\sqrt{(1+\beta^2)}} =$$

$$\frac{-(\alpha+\delta)(m-\theta\beta)}{2\sqrt{1+\beta^2}}.$$

$$\tilde{\Omega}_{2|22} = \frac{-\big\langle (\nabla_{\tilde e_2} A_\xi)\,\tilde e_2,\ \tilde f_2\big\rangle}{\sqrt{(1+\lambda_2^2)^3}} = \frac{-\big\langle \theta m^2\,\tilde e_0 + (\alpha+\delta)(m-\theta\beta)\ \tilde e_2, -\tilde e_1\big\rangle}{(1+m^2)\sqrt{1+m^2}} = 0.$$

Summarizing, we get

$$\tilde{\Omega}_2 = \begin{pmatrix} 0 & \dfrac{m^2(m\beta-\theta)}{2\sqrt{(1+\beta^2)(1+m^2)}} & 0 \\ \dfrac{m^2(m\beta-\theta)}{2\sqrt{(1+\beta^2)(1+m^2)}} & 0 & \dfrac{(\alpha+\delta)(\theta\beta-m)}{2\sqrt{(1+\beta^2)}} \\ 0 & \dfrac{(\alpha+\delta)(\theta\beta-m)}{2\sqrt{(1+\beta^2)}} & 0 \end{pmatrix}.$$

Thus, for totally geodesic field ξ we have a unique possible solution

$$\beta = \theta m, \quad m\beta = \theta.$$

It follows, then,

$$-\alpha\delta = m^2 = 1, \quad \beta = \theta.$$

As a consequence,

$$\pm\xi = \theta\frac{1}{\sqrt{1+\alpha^2}}\,e_2 + \frac{\alpha}{\sqrt{1+\alpha^2}}$$

is the corresponding totally geodesic unit vector field.

■

3.1 Geometrical description of totally geodesic unit vector field and the group

Proposition 3.1 *Let G be a non-unimodular three-dimensional Lie group with a left-invariant metric. Suppose G admits a left-invariant totally geodesic unit vector field ξ. Then either*

- *$G = L^2(-\alpha^2) \times E^1$, where $L^2(-\alpha^2)$ is the Lobachevski plane of curvature $-\alpha^2$, and ξ is a parallel unit vector field on G tangent to Euclidean factor, or*

- *G admits the Sasakian structure; moreover, G admits two hyperfoliations $\mathcal{L}_1, \mathcal{L}_2$ such that*

  (i) the foliations $\mathcal{L}_1$ and $\mathcal{L}_2$ are intrinsically flat, mutually orthogonal and has a constant extrinsic curvature,

  (ii) one of them, say $\mathcal{L}_2$, is minimal,

  (iii) the integral trajectories of the field ξ are $\mathcal{L}_1 \cap \mathcal{L}_2$.

Proof. Suppose ξ is as in the hypothesis. Consider the case $\beta = \delta = 0$ and $\xi = e_3$ of the Theorem 3.2. The bracket operations take the form

$$[e_1, e_2] = \alpha\, e_2, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0$$

and we conclude that the group admits three integrable distributions, namely, $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_2 \wedge e_3$. The Table of the Levi-Civita connection takes the form

∇	e_1	e_2	e_3
e_1	0	0	0
e_2	$-\alpha\, e_2$	$-\alpha\, e_1$	0
e_3	0	0	0

The only non-zero component of the curvature tensor of the group is of the form

$$R(e_1, e_2)e_2 = -\alpha^2 e_1.$$

Thus, $G = L^2(-\alpha) \times R^1$ and the field $\xi = e_3$ is a parallel unit vector field on G tangent to the Euclidean factor.

Consider the second case of the Theorem 3.2. If $\beta = \theta, m = \sqrt{-\alpha\,\delta} = 1$, then with respect to the singular frame the matrix A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and hence, $\xi = \theta\, \tilde{e}_0$ is the Killing unit vector field. Therefore, by the Theorem 2.3, the structure

$$(\phi = A_\xi,\ \xi,\ \eta = \langle \xi, \cdot \rangle);$$

is Sasakian.

We can also say more about this Sasakian structure. The Table (27) in the case under consideration takes the form

∇	$\tilde e_0$	$\tilde e_1$	$\tilde e_2$
$\tilde e_0$	0	$-\theta\,\tilde e_2$	$\theta\,\tilde e_1$
$\tilde e_1$	$-\theta\,\tilde e_2$	0	$\theta\,\tilde e_0$
$\tilde e_2$	$\theta\,\tilde e_1$	$-\theta\,\tilde e_0-(\alpha+\delta)\,\tilde e_2$	$(\alpha+\delta)\,\tilde e_1$

(29)

an hence, for the brackets we have

$$[\,\tilde e_0,\,\tilde e_1]=0,\quad [\,\tilde e_0,\,\tilde e_2]=0,\quad [\,\tilde e_1,\,\tilde e_2]=2\theta\,\tilde e_0+(\alpha+\delta)\,\tilde e_2. \tag{30}$$

From (30) we see that the distributions $\tilde e_0\wedge\tilde e_2$ and $\tilde e_0\wedge\tilde e_1$ are integrable. Denote by $\mathcal{L}_1$ and $\mathcal{L}_2$ the corresponding foliations generated by these distributions. Then the integral trajectories of the field ξ are exactly $\mathcal{L}_1\cap\mathcal{L}_2$.

Denote $\Omega^{(1)}$ and $\Omega^{(2)}$ a second fundamental form of the $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Since $\tilde e_1$ and $\tilde e_2$ are the unit normal vector fields for the corresponding foliations, from (29) we can easily find

$$\Omega^{(1)}=\begin{pmatrix}0 & 1\\ 1 & \alpha+\delta\end{pmatrix},\qquad \Omega^{(2)}=\begin{pmatrix}0 & -\theta\\ \theta & 0\end{pmatrix}$$

and see that $\mathcal{L}_2$ is a minimal foliation.

Setting $\xi=\theta\,\tilde e_0$, we can find from (28) the corresponding curvature components

$$R(\,\tilde e_0,\,\tilde e_2)\,\tilde e_0=-\,\tilde e_2,\quad R(\,\tilde e_0,\,\tilde e_1)\,\tilde e_0=-\,\tilde e_1.$$

Denote by $K^{(i)}_{int}$ and $K^{(i)}_{ext}$ the intrinsic and extrinsic curvatures of the corresponding foliations ($i=1,2$). Then $K^{(i)}_{ext}=\big\langle R(\,\tilde e_0,\,\tilde e_i)\,\tilde e_i,\,\tilde e_0\big\rangle=1$. The Gauss equation implies

$$K^{(i)}_{int}=K^{(i)}_{ext}+\det\Omega^{(i)}=0.$$

Therefore, both of the foliations are *intrinsically flat* and have a *constant extrinsic curvature* $K^{(i)}_{ext}=1$.

$\blacksquare$

4 Appendix

The structure (20) appears also in a different setting. Remind that the unit tangent bundle T_1M^n is a hypersurface in TM^n with a unit normal vector ξ^v at each point $(q,\xi)\in T_1M^n$. Define a unit vector field $\bar\xi$, a 1-form $\bar\eta$ and a $(1,1)$ tensor field $\bar\varphi$ on T_1M^n by

$$\bar\xi=-J\xi^v=\xi^h,\quad JX=\bar\varphi X+\bar\eta(X)\xi^v,$$

where J is a natural almost complex structure on TM^n, acting as

$$JX^v = -X^h, \quad JX^h = X^v.$$

The triple $(\bar{\xi}, \bar{\eta}, \bar{\varphi})$ form a standard almost contact structure on T_1M^n with Sasaki metric g_S. This structure is not almost contact *metric* one. By taking

$$\tilde{\xi} = 2\bar{\xi} = 2\xi^h, \quad \tilde{\eta} = \frac{1}{2}\bar{\eta}, \quad \tilde{\varphi} = \bar{\varphi}, \quad g_{cm} = \frac{1}{4}g_S$$

at each point $(q, \xi) \in T_1M^n$, we get the *almost contact metric structure* $(\tilde{\xi}, \tilde{\eta}, \tilde{\varphi})$ on (T_1M^n, g_{cm}).

In a case of a general almost contact metric manifold $(\tilde{M}, \tilde{\xi}, \tilde{\eta}, \tilde{\varphi}, \tilde{g})$ a submanifold N is called *invariant* if $\tilde{\varphi}(T_pN) \subset T_pN$ and *anti-invariant* if $\tilde{\varphi}(T_pN) \subset (T_pN)^\perp$ for every $p \in N$.

A unit vector field ξ on a Riemannian manifold (M^n, g) is called invariant (anti-invariant) is the submanifold $\xi(M^n) \subset (T_1M^n, g_{cm})$ is invariant (anti-invariant). Recently, Binh T.Q., Boeckx E. and Vanhecke L. have considered this kind of unit vector fields and proved the following Theorem [2].

Theorem 4.1 *A unit vector field ξ on (M^n, g) is invariant if and only if*

$$\left(\phi = A_\xi, \, \xi \, , \, \eta = \left\langle \xi, \cdot \right\rangle_g \right)$$

is an almost contact structure on M^n. In particular, ξ is a geodesic vector field on M^n and $n = 2m + 1$.

Summarizing the results of this section, we come to the following conclusion.

Proposition 4.1 *Every left invariant non-parallel totally geodesic unit vector field on a three-dimensional Lie group G with a left-invariant metric generates the invariant submanifold in (T_1G, g_{cm}).*

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